p-GROUP ACTIONS AND COBORDISM

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ABSTRACT. These are notes for a mini-course given at the University of Regensburg in September 2022 during the Summer school "Motives in Ratisbona". The subject is the study of actions of finite p-groups on algebraic varieties.

We begin by discussing various fixed point theorems, present methods to prove them, and illustrate them by applications and examples. Among the numerical invariants used to detect fixed points are the Chern numbers, whose consideration leads us to consider the cobordism ring.

We then provide a relatively self-contained account of the construction of the algebraic cobordism ring (following an elementary approach due to Merkurjev), and finally illustrate how this ring can be used to interpret the fixed point theorems, and permits to generalise them.

As prerequisites we assume familiarity with basic algebraic geometry, the Chow group and K-theory (only K_0). These notes contain no new results, but we attempt to explain in simple terms existing results and gather adequate references.

1. Prologue

In this section, we discuss methods for detecting fixed points using the Euler number. We refer to the papers [Ser09] and [EN11] for more details and discussions of related questions.

Let p be a prime number. A finite group is called a p-group if its cardinality is a power of p. Recall the following basic fact about p-groups:

1.1. **Proposition.** *Let S be a finite set equipped with an action of a p-group G. If the cardinality $|S|$ is prime to p, then the fixed subset S^G is nonempty.*

We would like to find a generalisation of this fact to algebraic varieties, instead of finite sets.

Let us fix a base field k, and call a quasi-projective scheme over k a *variety*. An *action* of a group G on a variety X will mean a group morphism $G \to \mathrm{Aut}_k(X)$. The *fixed locus* X^G is a closed subscheme of X such that $\mathrm{Hom}_k(Y, X^G) = \mathrm{Hom}_k(Y, X)^G$ for any variety Y (it may be defined as the equaliser of the morphisms induced by the actions of all elements of G). We say that G acts *freely* on X if $X^H = \varnothing$ for all subgroups $H \neq 1$ of G.

Perhaps the most faithful generalisation of the cardinality of finite sets to higher-dimensional varieties is the so-called Euler number:

1.2. **Definition.** The *Euler number* of a variety X is defined as

$$\chi(X) = \sum_{i=0}^{2 \dim X} (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{ét,c}}(X_{\overline{k}}; \mathbb{Q}_\ell) \in \mathbb{Z}.$$

Here $\overline{k}$ denotes an algebraic closure of k, and $H^i_{\text{ét,c}}(X_{\overline{k}};\mathbb{Q}_\ell)$ the compactly supported ℓ-adic cohomology groups of the $\overline{k}$-variety $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, where ℓ is a prime number invertible in k (see e.g. [Mil80])

1.3. **Example.** If $\mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$ denotes the n-dimensional affine space over k, we have $\chi(\mathbb{A}^n) = 1$, because

$$H^i_{\text{ét,c}}(\mathbb{A}^n_{\overline{k}};\mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{if } i = 2n, \\ 0 & \text{if } i \neq 2n. \end{cases}$$

1.4. **Proposition.** *We have*

(i) If Y is closed in X, we have

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y).$$

(ii) $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Proof (sketch). The first statement is a consequence of the long exact localisation sequence for compactly supported ℓ-adic cohomology groups [Mil80, III, Remark 1.30], and the second follows from Künneth theorem [Mil80, VI, Theorem 8.5]. □

1.5. **Example.** We have $\chi(\mathbb{P}^n) = n + 1$.

Observe that if $f : Y \to X$ is a Zariski-local fibration with fiber F (by this we means that X admits a covering by open subschemes U such that $f^{-1}U \simeq F \times U$ over U), then it follows from Proposition 1.4 that

$$\chi(Y) = \chi(X)\chi(F).$$

This permits to compute the Euler number of vector bundles, projective bundles or blow-ups.

The case of étale locally trivial fibrations is more subtle, at least in positive characteristic:

1.6. **Example.** Let k be a field of characteristic $p > 0$, and consider the morphism $\mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^p - x$. This morphism is étale of degree p, but as $\chi(\mathbb{A}^1) = 1$ we have $\chi(\mathbb{A}^1) \neq p\chi(\mathbb{A}^1)$.

When a group G acts on a variety X, the variety Y such that $\operatorname{Hom}_k(X, T)^G = \operatorname{Hom}_k(Y, T)$ for any variety T, if it exists, is called the *G-quotient* of X and denoted by X/G. The G-quotient always exists when G is finite (see e.g. [SGA1, V, Proposition 1.8] or [SGA3-1, V, Théorème 4.1(i)], the key point being that our varieties are quasi-projective). When $X = \operatorname{Spec} A$ is affine, then G acts on the k-algebra A and $X/G = \operatorname{Spec}(A^G)$.

1.7. **Proposition.** *Let G be a finite group of order prime to the characteristic of k. Assume that G acts freely on X. Then*

$$\chi(X) = |G| \cdot \chi(X/G).$$

Proof. See [IZ13, §3]. The arguments can already be found in [Ver73] in the topological setting. The idea is the following: we refine $\chi(X) \in \mathbb{Z}$ to a virtual G-representation over the field $\mathbb{Q}_\ell$, and show that it is a multiple of the regular representation, by computing its character using Lefschetz trace formula. □

1.8. **Corollary** ([Ser09, §7.2]). *Assume that p is unequal to the characteristic of k. Let G be a p-group acting on X. If $\chi(X)$ is prime to p, then $X^G \neq \varnothing$.*

Proof. Induction on $|G|$. If $G \neq 1$, then there exists a central subgroup H isomorphic to $\mathbb{Z}/p$. The group H then acts freely on $U = X \smallsetminus X^H$, hence $\chi(U)$ is divisible by p, and so $\chi(X^H) = \chi(X) - \chi(U)$ is prime to p. By induction applied to the G/H-action on X^H, we deduce that $X^G = (X^H)^{G/H} \neq \varnothing$. □

1.9. **Example.** Since $\chi(\mathbb{A}^n) = 1$, we see that $(\mathbb{A}^n)^G \neq \varnothing$ for every action of a p-group G on the affine space $\mathbb{A}^n$ over a field k of characteristic unequal to p. Note that this fails when k has characteristic p: indeed the translation by any nonzero vector yields an action of $\mathbb{Z}/p$ on $\mathbb{A}^n$ having no fixed point.

The next lemma yields an effective way of computing the Euler number in certain cases:

1.10. **Lemma.** *Assume that X is smooth and projective of pure dimension d. Then*

$$\chi(X) = \deg c_d(T_X),$$

where T_X is the tangent bundle of X, and $c_d(T_X)$ its d-th Chern class with values in the Chow ring $\mathrm{CH}(X)$.

Proof. This follows from the Lefschetz trace formula [Mil80, VI, Theorem 12.3] and the self intersection formula [Ful98, Example 8.1.12]. □

From this we deduce:

1.11. **Lemma.** *Assume that k has characteristic zero. Let X be a variety. Then X supports a zero-cycle of degree $\chi(X)$.*

Proof. This follows from resolution of singularities and a moving lemma, see [Hau17, Proposition 3.1.4] for details (and a generalisation to positive characteristic). □

In particular if a p-group G acts on X with $\chi(X)$ prime to p over k of characteristic zero, the variety X^G supports a zero-cycle of degree prime to p.

1.12. **Remark.** This last fact holds more generally when k has characteristic unequal to p.

When one has information on the étale cohomology groups of X (as opposed to knowing just the number $\chi(X)$), more precise methods can be used, for instance:

(i) *Lefschetz fixed point theorem* (see [Ser09, §7.3]): if G is cyclic of order not divisible by the characteristic of k, generated by g

$$\chi(X^G) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(g : H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell)).$$

In particular when $X = \mathbb{A}^n$, then $X^G \neq \varnothing$ for such groups G.

(ii) *Smith theory* (see [Ser09, §7.4]): Let us call a variety Y *p-acyclic* if $H^0_{\mathrm{et}}(Y_{\overline{k}}, \mathbb{F}_p) = \mathbb{F}_p$ and $H^i_{\mathrm{et}}(Y_{\overline{k}}, \mathbb{F}_p) = 0$ for $i > 0$. Assume that G is a p-group acting on X. Then if X is p-acyclic, so is X^G. In particular X^G is geometrically connected. This applies for instance when $X = \mathbb{A}^n$.

The special case $X = \mathbb{A}^n$ is particularly rich, and can be approached with a variety of methods. For instance, it is known that $(\mathbb{A}^n)^G \simeq \mathbb{A}^s$ for some s when $n \leqslant 2$ and G is a p-group acting on $\mathbb{A}^n$ with $p \neq \operatorname{char} k$, but this is an open question for larger n. We refer to the survey [Kra96] for a discussion of open problems concerning the automorphism group of the affine space (called the *affine Cremona group*).

2. Fixed point theorems

2.1. Chern numbers.

2.1.1. **Definition.** Let X be a smooth projective variety. To a collection $(i_1, \ldots, i_n) \in \mathbb{N}^n$ corresponds the *Chern number*

$$\deg(c_{i_1}(T_X) \cdots c_{i_n}(T_X)) \in \mathbb{Z}.$$

Here T_X denotes the tangent bundle of X, and the Chern classes $c_{i_j}(T_X)$ take values in the Chow ring $\operatorname{CH}(X)$.

In particular the Euler number $\chi(X)$ considered in the previous section is a Chern number by Lemma 1.10. It might be desirable to detect fixed points using other Chern numbers, since there are many Chern numbers as opposed to a single Euler number. There are however some difficulties:

- We must restrict ourselves to (smooth) projective varieties in order to consider the Chern numbers.
- This tends to breaks the inductive arguments, even if we start with a projective variety: if X is projective, so is X^G, but not $X \smallsetminus X^G$!
- Indeed, we will have to restrict ourselves to the consideration of certain types of p-groups.

Some results in this direction are gathered in the following statement:

2.1.2. **Theorem** ([Hau19, (1.1.1)]). *Let X be a smooth projective variety with an action of a p-group G. Assume that one of the following conditions holds:*

(i) *G is abelian.*
(ii) $\operatorname{char} k = p$.
(iii) $\dim X < p$.

If a Chern number of X is prime to p, then $X^G \neq \varnothing$.

2.1.3. **Remark.** When $p = 2$, the last condition can be replaced with $\dim X < 4$. This is because when $\dim X = 2$ the two Chern numbers of X have the same parity (and one of them is $\chi(X)$), while when $\dim X = 3$ all Chern numbers are even. With this change, the statement of Theorem 2.1.2 becomes quite sharp (examples will be discussed later).

2.1.4. **Remark.** It follows from Theorem 2.1.2 that Corollary 1.8 actually holds in characteristic p, *provided that X is projective.*

Let us briefly discuss some elements of the proof of Theorem 2.1.2. The case $G = \mathbb{Z}/p$ is easy, because of the following:

2.1.5. **Lemma.** *Let G be a finite group, and X be a smooth projective variety with a free G-action. Then $Y = X/G$ is smooth, and for any $i_1, \ldots, i_n \in \mathbb{N}$*

$$\deg(c_{i_1}(T_X) \cdots c_{i_n}(T_X)) = |G| \cdot \deg(c_{i_1}(T_Y) \cdots c_{i_n}(T_Y))$$

Proof. Indeed, the quotient morphism $\pi\colon X \to Y$ is étale, hence Y is smooth and $T_X = \pi^* T_Y$. Since π has degree $|G|$, we have $\pi_*(1) = |G|$ in $\mathrm{CH}(Y)$, and it follows from the projection formula that we have in $\mathrm{CH}(Y)$

$$\pi_*(c_{i_1}(T_X)\cdots c_{i_n}(T_X)) = \pi_* \circ \pi^*(c_{i_1}(T_Y)\cdots c_{i_n}(T_Y)) = |G| \cdot c_{i_1}(T_Y)\cdots c_{i_n}(T_Y),$$

and we conclude by taking degrees. □

Assume now that G is a finite group, and X a variety with a G-action. We consider the equivariant Chow ring $\mathrm{CH}_G(X)$, obtained using an algebraic version of Borel's construction (see e.g. [EG98]). There is a forgetful morphism $\mathrm{CH}_G(X) \to \mathrm{CH}(X)$, and the Chern classes of G-equivariant vector bundles lift to $\mathrm{CH}_G(X)$. In particular, when X is smooth and projective, its Chern numbers are in the image of the composite $\mathrm{CH}_G(X) \to \mathrm{CH}(X) \xrightarrow{\deg} \mathbb{Z}$.

In general, the elements of $\mathrm{CH}_G(X)$ are not represented by G-invariant closed subschemes of X, but rather by $\mathbb{Z}$-linear combination of G-invariant closed subschemes of $X \times V$, where V runs over the finite-dimensional G-representations. However:

2.1.6. **Lemma.** *Assume that G is trigonalisable over k, i.e. every G-representation over k admits a subrepresentation of codimension one. Then the image of $\mathrm{CH}_G(X) \to \mathrm{CH}(X)$ is the subgroup generated by classes of G-invariant closed subschemes of X.*

Sketch of proof. For a variety Y, denote by $\mathcal{Z}(Y)$ the group of cycles on Y, and if G acts on Y by $\mathcal{Z}_G(Y) \subset \mathcal{Z}(Y)$ the subgroup generated by classes of G-invariant closed subschemes. Let V be a finite-dimensional G-representation. Then the composite

$$\mathcal{Z}_G(X \times V) \to \mathrm{CH}_G(X) \to \mathrm{CH}(X)$$

coincides with

$$\mathcal{Z}_G(X \times V) \to \mathcal{Z}(X \times V) \to \mathrm{CH}(X \times V) \to \mathrm{CH}(X).$$

Let $V' \subset V$ be a codimension one subrepresentation. Then one shows that the image of

$$\mathcal{Z}_G(X \times V) \to \mathcal{Z}(X \times V) \to \mathrm{CH}(X \times V) \to \mathrm{CH}(X \times V')$$

is contained in the image of

$$\mathcal{Z}_G(X \times V') \to \mathcal{Z}(X \times V') \to \mathrm{CH}(X \times V')$$

(using the explicit description of the intersection with a divisor in the Chow group, and the fact that $X \times V' \subset X \times V$ is a principal divisor), and we conclude by induction on $\dim V$. □

To prove Theorem 2.1.2, we may assume that k is algebraically closed. Then under the condition (i) or (ii), the group G is trigonalisable over k, hence by Lemma 2.1.6 and the assumption on the Chern numbers of X, there exists a G-invariant closed subscheme in X having degree prime to p. Since G is a p-group, this is only possible if $X^G \neq \varnothing$.

2.2. Coherent Euler characteristics.

2.2.1. **Example.** Assume that X is a smooth connected projective variety of dimension 1. Then the only Chern number of X is

$$\deg c_1(T_X) = 2(1-g),$$

where g is the geometric genus of X. In particular all Chern numbers of X are even, and therefore Theorem 2.1.2 becomes empty for such X when $p = 2$.

In the situation of Example 2.2.1, we would rather like to use the parity of the genus g to detect fixed points of 2-groups. Let us first generalise the genus of curves to higher dimensional varieties:

2.2.2. **Definition.** Let X be a projective k-variety. The Euler characteristic of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is defined as the integer

$$\chi(X,\mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X,\mathcal{F}) \in \mathbb{Z}. \tag{2.2.2.a}$$

In particular we obtain an invariant $\chi(X,\mathcal{O}_X) \in \mathbb{Z}$. When X is smooth and projective, this invariant is not a $\mathbb{Z}$-linear combination of Chern numbers. However we have the Hirzebruch–Riemann–Roch formula

$$\chi(X,\mathcal{O}_X) = \deg \operatorname{Td}(T_X) \in \mathbb{Q},$$

where $\operatorname{Td}(T_X) \in \operatorname{CH}(X) \otimes \mathbb{Q}$ is the so-called Todd class. So $\chi(X,\mathcal{O}_X)$ is a $\mathbb{Q}$-linear combination of Chern numbers, which happens to take integral values on all smooth projective varieties X.

2.2.3. **Proposition.** *The integer $\chi(X,\mathcal{O}_X)$ is a birational invariant of the smooth projective variety X.*

Proof. In fact, the groups $H^i(X,\mathcal{O}_X)$ themselves are birational invariants, see e.g. [CR11, Theorem 3.2.8]. □

2.2.4. **Proposition.** *When X, Y are smooth projective varieties, we have*

$$\chi(X \times Y, \mathcal{O}_{X\times Y}) = \chi(X,\mathcal{O}_X) \cdot \chi(Y,\mathcal{O}_Y).$$

Proof. See e.g. [Ful98, Example 15.2.12]. □

2.2.5. **Theorem** ([Hau19, (1.2.1)])**.** *Let X be a projective variety with an action of a p-group G. Assume that one of the following conditions holds:*

(i) *G is cyclic.*
(ii) $\operatorname{char} k = p$.
(iii) $\dim X < p-1$.

If $\mathcal{F}$ is a G-equivariant coherent $\mathcal{O}_X$-module such that $\chi(X,\mathcal{F})$ is prime to p, then $X^G \neq \varnothing$.

2.2.6. **Remark.** The $\mathcal{O}_X$-module $\mathcal{O}_X$ itself is G-equivariant. Note that $\chi(X,\mathcal{O}_X) = 1$ when X is a geometrically connected and $H^i(X,\mathcal{O}_X) = 0$ for $i > 0$. This happens for instance X is rational, or $\operatorname{char} k = 0$ and X is rationally connected (which means that two general points are contained in a rational curve). In this case, the result (i) of Theorem 2.2.5 is a consequence of a Lefschetz fixed-point theorem for coherent cohomology, and (ii) follows from Smith theory.

The next examples illustrate the sharpness of the conditions:

2.2.7. **Example.** Assume that $\operatorname{char} k \neq 2$. Consider the involutions of $\mathbb{P}^1$ given by $\sigma\colon [x:y] \mapsto [y:x]$ and $\tau\colon [x:y] \mapsto [-x:y]$. These involutions commute with one another, giving an action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ on $\mathbb{P}^1$. The fixed points of σ are $[1:1], [1:-1]$, and those of τ are $[1:0], [0:1]$, hence G has no fixed point on $\mathbb{P}^1$. On the other hand, we have $\chi(X,\mathcal{O}_X) = 1$.

2.2.8. **Example.** Let G be a nonabelian p-group, where k has characteristic $\neq p$. Then there exists an irreducible G-representation V of dimension p^n with $n > 0$. When G has order p^3, then $n = 1$. The k-variety $X = \mathbb{P}(V)$ with its induced G-action satisfies $X^G = \varnothing$. We have $\chi(X, \mathcal{O}_X) = 1$ and $\dim X = p^n - 1$. The blow-up Y of $\mathbb{P}(V \oplus 1)$ at the point $\mathbb{P}(1)$ has a natural G-action without fixed points, and a Chern number of Y is prime to p when $p \neq 2$ (namely $\deg(c_1(T_Y)^{p^n})$).

The proof of the next statement illustrates how the existence of fixed points can be used to prove properties of automorphisms groups of varieties:

2.2.9. **Corollary** ([Xu20])**.** *Assume that k is algebraically closed of characteristic zero. Let X be a rationally connected variety of dimension n. Assume that the group of birational automorphisms $\mathrm{Bir}(X)$ contains a subgroup G, which is a p-group for $p > n+1$. Then G is abelian of rank at most n.*

Proof (sketch). Using equivariant resolution of singularities, we may assume that X is smooth projective and that $G \subset \mathrm{Aut}(X)$. This procedure does not change the fact that X is rationally connected, and so we have $\chi(X, \mathcal{O}_X) = 1$. Then X admits a G-fixed point x by Theorem 2.2.5. One may then prove that the G-action on the tangent space $T_{X,x}$ is faithful (using the fact that G is finite and k has characteristic zero, see [Pop14, Lemma 4]). Since $p > n = \dim_k T_{X,x}$ and G is a p-group, we deduce that $T_{X,x}$ contains no irreducible representation of dimension > 1 (irreducible representations of p-groups have dimension a power of p). Therefore G has a faithful representation of dimension n, which is a direct sum of 1-dimensional representations, and the result follows. □

2.3. **Grothendieck groups.** Let us now discuss the proof of (i) and (iii) in Theorem 2.2.5. When X is a variety, we denote by $K_0'(X)$ the Grothendieck group of coherent $\mathcal{O}_X$-modules. It is defined as the quotient of the free abelian group on classes of coherent $\mathcal{O}_X$-modules, modulo the relations $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ whenever

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of coherent $\mathcal{O}_X$-modules. If $f \colon Y \to X$ is a flat morphism, then pulling back coherent $\mathcal{O}_X$-modules along f induces a group morphism $f^* \colon K_0'(X) \to K_0'(X)$. If $f \colon Y \to X$ is a projective morphism (not necessarily flat), setting

$$f_*[\mathcal{F}] = \sum_i (-1)^i [R f_*^i \mathcal{F}]$$

induces a push-forward map $f_* \colon K_0'(Y) \to K_0'(X)$. Note that, when $p \colon X \to \operatorname{Spec} k$ is projective, we have $p_*[\mathcal{F}] = \chi(X, \mathcal{F}) \in K_0'(\operatorname{Spec} k) = \mathbb{Z}$.

As above, the case $G = \mathbb{Z}/p$ is easy:

2.3.1. **Lemma.** *Let X be a projective variety with a free G-action, and denote by $\pi \colon X \to Y = X/G$ the G-quotient. Then for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ we have*

$$\chi(X, \pi^*\mathcal{F}) = |G| \cdot \chi(Y, \mathcal{F}).$$

Proof. We proceed by induction on the dimension of the support of $\mathcal{F}$. Since the group $K_0'(Y)$ is generated by classes $[\mathcal{O}_Z]$, where $Z \subset Y$ is a closed subscheme, replacing X with $X \times_Y Z$, we may assume that $\mathcal{F} = \mathcal{O}_Y$. Let $d = |G|$. As $d \neq 0$, it will suffice to prove the statement for $\mathcal{F} = (\mathcal{O}_Y)^{\oplus d}$ instead. Then $[\pi_* \mathcal{O}_X] - [(\mathcal{O}_Y)^{\oplus d}] \in K_0'(Y)$ belongs to the subgroup generated by classes of $\mathcal{O}_Y$-modules supported in codimension at least one, hence by induction it will suffice to prove the statement for $\mathcal{F} = \pi_* \mathcal{O}_X$.

As the G-action on X is free, we have a cartesian square

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array}$$

where a is the action morphism and p the projection to the second factor. We view G as scheme, the disjoint union of d copies of $\operatorname{Spec} k$. Then

$$\pi^* \pi_* \mathcal{O}_X \simeq p_* a^* \mathcal{O}_X = p_* \mathcal{O}_{G \times X} = (\mathcal{O}_X)^{\oplus d},$$

hence $\chi(X, \pi^*\pi_*\mathcal{O}_X) = d \cdot \chi(X, \mathcal{O}_X) = d \cdot \chi(Y, \pi_*\mathcal{O}_X$, as required. □

Let G be a finite group and X a projective variety with a G-action. Proceeding as above, we define the Grothendieck group of G-equivariant coherent $\mathcal{O}_X$-modules $K_0'(X; G)$. We let p be a prime number, and define

(2.3.1.a) $$d_p(X;G) = \operatorname{im}(K_0'(X;G) \to K_0'(X) \xrightarrow{\chi(X,-)} \mathbb{Z} \to \mathbb{F}_p).$$

The following observation will be crucial:

2.3.2. **Lemma.** *Let $H \subset G$ a central subgroup acting trivially on X. Assume that k is algebraically closed. Then $d_p(X;G) = d_p(X;G/H)$.*

Proof. We let $\mathcal{F}$ be a G-equivariant coherent $\mathcal{O}_X$-module such that $\chi(X, \mathcal{F})$ is prime to p, and show that there exists a G/H-equivariant coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that $\chi(X, \mathcal{G})$ is prime to p. As $H \subset G$ is central and k contains enough roots of unity, the $\mathcal{O}_X$-module $\mathcal{F}$ admits a G-equivariant decomposition $\mathcal{F} = \bigoplus_h \mathcal{F}_h$, where h runs over the characters of H, and H acts on $\mathcal{F}_h$ via h. We may thus assume that H acts on $\mathcal{F}$ through a single character h. As $\chi(X, \mathcal{F})$ is prime to p, there is an integer i such that $\dim_k H^i(X, \mathcal{F})$ is prime to p. Set $V = H^i(X, \mathcal{F})$. Then H acts trivially on $\mathcal{G} = \mathcal{F} \otimes V^\vee$, and $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) \cdot \dim_k V^\vee$ is prime to p. □

2.3.3. **Lemma.** *Let G be a cyclic p-group, and $H \subset G$ a subgroup such that $H \neq G$. Let X be a variety with an action of G, and set $Y = X/H$. Then the morphism $X^G \to Y^G$ is surjective.*

Proof. Let K be an algebraically closed field, and consider $y \in Y^G(K)$. Since the quotient $X \to Y$ is surjective (see [SGA1, V, Proposition 1.1 (ii)]), we may find $x \in X(K)$ mapping to y. Let $g \in G$ be a generator. As $g \cdot y = y$, we may find $h \in H$ such that $g \cdot x = h \cdot x$ (the fibers of $X \to Y$ are the H-orbits, see [SGA1, V, Proposition 1.1 (ii)]). But $h = g^{pu}$ for some $u \in \mathbb{Z}$, hence x is fixed by $gh^{-1} = g^{1-pu}$, which is a generator of G. Thus $x \in X^G(K)$, as required. □

We can now prove (i) in Theorem 2.2.5: If $G \neq 1$, let $H \subset G$ be the subgroup of index p. Consider the quotient morphism $\varphi \colon X \to Y = X/H$. If $X^G = \varnothing$, then $Y^G = Y^{G/H} = \varnothing$ by the Lemma 2.3.3. If $d_p(X;G) \neq 0$, then $d_p(Y;G) \neq 0$, hence $d_p(Y;G/H) \neq 0$ by Lemma 2.3.2. We may thus replace X with Y and G with G/H, and thus assume that $G \simeq \mathbb{Z}/p$. This case was treated in Lemma 2.3.1.

We now discuss the proof of (iii) in Theorem 2.2.5. Let G be a finite group and X a variety with a G-action. Besides $K_0'(X;G)$, one may also consider an equivariant theory constructed using Borel's construction (similar to the equivariant Chow ring $\mathrm{CH}_G(X)$ mentioned above). However will be more useful to consider a simpler theory:

2.3.4. **Definition.** Let V be the regular representation of G over k. We let $T = \operatorname{Spec} k(V)$, and consider the field $K = k(V)^G$. Viewing the scheme $(X \times T)/G$ as a K-variety, we set

$$K_G(X) = K_0'((X \times T)/G).$$

2.3.5. **Remark.** In the definition of $K_G(X)$, we may replace V with any finite-dimensional, generically free G-representation, and obtain a canonically isomorphic group $K_G(X)$.

2.3.6. **Proposition.** *The forgetful morphism factors as* $K_0'(X;G) \to K_G(X) \to K_0'(X)$, *where the morphism* $K_G(X) \to K_0'(X)$ *is surjective.*

Proof. We have morphisms (recall T is the spectrum of a purely transcendental extension)

$$K_G(X) = K_0'((X \times T)/G) \to K_0'(X \times T) \simeq K_0'(X).$$

and

$$K_0'(X;G) \xrightarrow{\alpha} K_0'(X \times V;G) \xrightarrow{\beta} K_0'(X \times T;G) \simeq K_0'((X \times T)/G) = K_G(X),$$

whose composite is the canonical morphism $K_0'(X;G) \to K_0'(X)$. Here α is surjective by equivariant homotopy invariance, and β is so by the equivariant localisation sequence. □

2.3.7. **Remark.** Observe that the morphism $K_G(\operatorname{Spec} k) \to K_0'(\operatorname{Spec} k) = \mathbb{Z}$ is bijective, so in a sense K_G is much closer to K_0' than is $K_0'(-;G)$.

Note that Proposition 2.3.6 implies that

$$d_p(X;G) = \operatorname{im}(K_G(X) \to K_0'(X) \xrightarrow{\chi(X,-)} \mathbb{Z} \to \mathbb{F}_p). \tag{2.3.7.a}$$

2.3.8. **Proposition.** *The group* $K_G(X)$ *is endowed with a filtration*

$$\cdots \subset K_G(X)_{(n-1)} \subset K_G(X)_{(n)} \subset \dots$$

vanishing in negative degrees, which is compatible with the topological filtration of $K_0'(X)$ *(by codimension of supports), via the morphism* $K_G(X) \to K_0'(X)$.

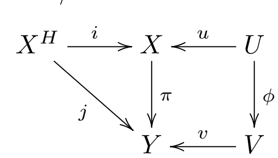
Proof. The filtration is induced by the topological filtration on $K_0'((X \times T)/G)$. □

The theory K_G has flat pullbacks and projective pushforwards along G-equivariant morphisms. In addition, it satisfies the localisation axiom:

2.3.9. **Lemma.** *Let* $i\colon Y \to X$ *be the immersion of a* G*-invariant closed subscheme, and* $u\colon U \to X$ *its open complement. Then the following sequence is exact:*

$$K_G(Y) \xrightarrow{i_*} K_G(X) \xrightarrow{u^*} K_G(U) \to 0.$$

Let us put ourselves in the situation of (iii) in Theorem 2.2.5. We assume that $\operatorname{char} k \neq p$ and $G \neq 1$. Then the group G contains a central subgroup H isomorphic to $\mathbb{Z}/p$. Let $Y = X/H$, $U = X \smallsetminus X^H$ and $V = U/H$. We thus have a commutative diagram



Here the square is cartesian, and i, j are closed immersions (as $\operatorname{char} k \neq p$).

2.3.10. **Lemma.** *The morphism $\phi^* \colon K_G(V) \to K_G(U)$ is surjective.*

Proof. Since the H-action on U is free, the category of G-equivariant coherent $\mathcal{O}_U$-modules is naturally equivalent to that of G/H-equivariant coherent $\mathcal{O}_V$-modules, inducing an isomorphism $K_0'(V; G/H) \to K_0'(U; G)$. This isomorphism factors through the morphism $\phi^* \colon K_0'(V;G) \to K_0'(U;G)$, hence the latter is surjective. Since $K_0'(U;G) \to K_G(U)$ is surjective by Proposition 2.3.6, the lemma follows. □

2.3.11. **Lemma.** *Assume that k contains a primitive p-th root of unity. Then*

$$\phi_* \circ \phi^*\big(K_G(V)\big) \subset pK_G(V).$$

Proof. The assumption implies that H is isomorphic as an algebraic group to μ_p; let us fix such an isomorphism. Then the G-equivariant $\mathcal{O}_V$-module $\phi_*\mathcal{O}_U$ is $\mathbb{Z}/p$-graded, and the fact that the G-action on U is free implies the existence of an invertible $\mathcal{O}_U$-module $\mathcal{L}$ and of a G-equivariant decomposition (recall that H is assumed to be central in G)

$$\phi_*\mathcal{O}_U = \bigoplus_{i=0}^{p-1} \mathcal{L}^{\otimes i},$$

Then we compute (as π_U is finite, the higher direct images $R^i\phi_*$ vanish for $i > 0$), for any $\beta \in K_G(V)$ (we use the projection formula)

$$\phi_* \circ \phi^*(\beta) = \sum_{i=0}^{p-1} [\mathcal{L}]^i \cap \beta,$$

where $\cap$ refers to the natural action of the Grothendieck group $K_0(V;G)$ of G-equivariant *locally free* coherent $\mathcal{O}_X$-modules on $K_G(V)$. Now, in $K_0(V;G)$ we have

$$\sum_{i=0}^{p-1} [\mathcal{L}]^i = (1 - [\mathcal{L}])^{p-1} + p \cdot Q([\mathcal{L}]),$$

for some polynomial $Q \in \mathbb{Z}[x]$. It is not difficult to see that $(1 - [\mathcal{L}]) \cap K_G(V)_{(n)} \subset K_G(V)_{(n-1)}$ for every n. As $\dim V \leqslant p-2$ by assumption and $K_G(V)_{(n)} = 0$ when $n < 0$ by Proposition 2.3.6, the statement follows. □

Consider now the commutative diagram

$$\begin{array}{ccccccc}
 & & K_G(X) & \xrightarrow{u^*} & K_G(U) & & \\
 & & \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \phi_*} & & \\
K_G(X^H) & \xrightarrow{j_*} & K_G(Y) & \xrightarrow{v^*} & K_G(V) & \longrightarrow & 0
\end{array}$$

where the lower row is an exact sequence by Lemma 2.3.9. A diagram chase using Lemma 2.3.10 and Lemma 2.3.11 then shows that

$$\pi_* K_G(X) = j_* K_G(X^H) \mod p.$$

Thus, in view of (2.3.7.a), it follows that (see (2.3.1.a))

$$d_p(X;G) = d_p(X^H;G).$$

We conclude the proof by induction on $|G|$, using Lemma 2.3.2.

3. The cobordism ring

3.1. **Oriented cohomology theories.** Recall that k is a field. We denote by Sm_k the category of smooth quasi-projective k-schemes.

3.1.1. **Definition** (Oriented cohomology theories, see [LM07, Definition 1.1.2])**.** A functor H from $\mathrm{Sm}_k^{\mathrm{op}}$ to the category of $\mathbb{Z}$-graded commutative rings, together with a group morphism $f_*^{\mathrm{H}}\colon \mathrm{H}(Y) \to \mathrm{H}(X)$ for each projective morphism $f\colon Y \to X$ in Sm_k, is called an *oriented cohomology theory* if the conditions (i)–(vii) below are satisfied. We write f^*_{H} instead of $\mathrm{H}(f)$ when f is a morphism in Sm_k, and denote by $\mathrm{H}^n(X)$ the degree n component of $\mathrm{H}(X)$.

(i) If $X, Y \in \mathrm{Sm}_k$ are connected and $f\colon Y \to X$ is a projective morphism, then f_*^{H} is homogeneous of degree $\dim X - \dim Y$.
(ii) ("Projection formula") If $f\colon Y \to X$ is a projective morphism in Sm_k, then $f_*^{\mathrm{H}}(af^*_{\mathrm{H}}(b)) = f_*^{\mathrm{H}}(a)b$ for any $a \in \mathrm{H}(Y), b \in \mathrm{H}(X)$.
(iii) ("Functoriality of push-fowards") If $X \in \mathrm{Sm}_k$, then $(\mathrm{id}_X)_* = \mathrm{id}_{\mathrm{H}(X)}$. If $f\colon Y \to X$ and $g\colon Z \to Y$ are projective morphisms in Sm_k, then $f_*^{\mathrm{H}} \circ g_*^{\mathrm{H}} = (f \circ g)_*^{\mathrm{H}}$.
(iv) ("Additivity") For any $X, Y \in \mathrm{Sm}_k$ the natural morphism $\mathrm{H}(X \sqcup Y) \to \mathrm{H}(X) \times \mathrm{H}(Y)$ is bijective.
(v) ("Base-change formula") Given a transverse square in Sm_k

$$\begin{array}{ccc} W & \xrightarrow{h} & Z \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

with f projective, we have

$$h_*^{\mathrm{H}} \circ e^*_{\mathrm{H}} = g^*_{\mathrm{H}} \circ f_*^{\mathrm{H}}.$$

(The transverse condition means that the square above is cartesian, and that for every connected component W_0 of W, denoting by Y_0, Z_0, X_0 the connected components of Y, Z, X containing the images of W_0,

$$\dim W_0 + \dim X_0 = \dim Y_0 + \dim Z_0.)$$

(vi) ("Projective bundle Theorem") Let E be a rank r vector bundle over $X \in \mathrm{Sm}_k$ and $p\colon \mathbb{P}(E) \to X$ the associated projective bundle. Denote by $s\colon \mathbb{P}(E) \to \mathcal{O}_{\mathbb{P}(E)}(1)$ the zero-section of the canonical bundle, and write $\xi = s^*_{\mathrm{H}} \circ s_*^{\mathrm{H}}(1) \in \mathrm{H}^1(\mathbb{P}(E))$. Then $1, \xi, \ldots, \xi^{r-1}$ is a basis of the $\mathrm{H}(X)$-module $\mathrm{H}(\mathbb{P}(E))$ (for the structure induced by p^*_{H}).
(vii) ("Homotopy invariance") Let $p\colon V \to X$ in Sm_k be an affine bundle (i.e. a torsor under a vector bundle over X). Then $p^*_{\mathrm{H}}\colon \mathrm{H}(X) \to \mathrm{H}(V)$ is bijective.

3.1.2. **Remark.** There is no "localisation axiom".

3.1.3. **Example.** The Chow ring CH is an example of oriented cohomology theory. Another example is given by the Grothendieck group of coherent modules. Indeed when $X \in \mathrm{Sm}_k$ the group $K_0'(X)$ may be identified with the Grothendieck group of locally free coherent $\mathcal{O}_X$-modules, which endows it with a ring structure, and permits to define pullbacks along arbitrary morphisms in Sm_k. For $X \in \mathrm{Sm}_k$ we set $K(X) = K_0'(X)[t, t^{-1}]$ where t has degree -1. Pullbacks in K are induced by those in K_0'. If $f\colon Y \to X$ is a projective morphism of pure codimension d in Sm_k, the push-forward $K(Y) \to K(X)$ is induced by

$$xt^i \mapsto f_*(x)t^{i-d}.$$

One then checks we thus define an oriented cohomology theory K.

For the rest of this section, we fix an oriented cohomology theory H.

3.1.4. **Definition.** Let V be a vector bundle of rank r over $X \in \mathrm{Sm}_k$. Using the notation of (3.1.1.vi) for $E = V^\vee$, the *Chern classes* $c_i^{\mathrm{H}}(V) \in \mathrm{H}^i(X)$ are defined using Grothendieck's method [Gro58] by setting

$$c_0^{\mathrm{H}}(V) = 1 \quad \text{and} \quad c_i^{\mathrm{H}}(V) = 0 \text{ if } i \notin \{0, \dots, r\},$$

and

$$\sum_{i=0}^{r} (-1)^i p_{\mathrm{H}}^*\big(c_i^{\mathrm{H}}(V)\big)\xi^{r-i} = 0 \in \mathrm{H}^r(\mathbb{P}(E)).$$

We will use the simplified notation f_*, f^*, c_i instead of $f_*^{\mathrm{H}}, f_{\mathrm{H}}^*, c_i^{\mathrm{H}}$ when no confusion seems likely to arise. If $j \colon Y \to X$ is a closed immersion in Sm_k, we will write $[Y] = j_*(1) \in \mathrm{H}(X)$.

3.1.5. **Remark.** When $L \to X$ is a line bundle, and $s \colon X \to L$ its zero-section, it follows from the definition that

$$c_1(L) = s^* \circ s_*(1) \in \mathrm{H}^1(X).$$

In particular, in the notation of (3.1.1.vi) we have $\xi = c_1(\mathcal{O}(1))$.

Assume now that $Z \in \mathrm{Sm}_k$ is the zero-locus in X of a section of L transverse to the zero-section $z \colon X \to L$ (in the sense of (3.1.1.v)). Then by homotopy invariance (3.1.1.vii), we have $z^* = s^* \colon \mathrm{H}(L) \to \mathrm{H}(X)$, hence by the base-change axiom (3.1.1.v), we deduce that

$$c_1(L) = [Z]. \tag{3.1.5.a}$$

In particular we have, for any $s \in \mathbb{N}$ (writing $\mathbb{P}^i = \varnothing$ for $i < 0$)

$$c_1(\mathcal{O}(1))^s = [\mathbb{P}^{n-s}] \in \mathrm{H}^s(\mathbb{P}^n), \tag{3.1.5.b}$$

where $\mathbb{P}^{n-s} \subset \mathbb{P}^n$ is any linear embedding.

3.1.6. (Splitting principle) Let E be a vector bundle of rank r over $X \in \mathrm{Sm}_k$. Consider the projective bundle $p \colon \mathbb{P}(E^\vee) \to X$. Then the pullback $p^* \colon \mathrm{H}(X) \to \mathrm{H}(\mathbb{P}(E^\vee))$ by the axiom (3.1.1.vi), and we have an exact sequence of vector bundles over $\mathbb{P}(E^\vee)$

$$0 \to U \to p^*E \to \mathcal{O}(1) \to 0,$$

where U is a vector bundle of rank $r-1$ over $\mathbb{P}(E^\vee)$. Iterating, we see that there exists a composite of projective bundles $q \colon P \to X$ such that q^*E admits a filtration by subbundles whose successive quotients are line bundles. Note that the pullback $q^* \colon \mathrm{H}(X) \to \mathrm{H}(P)$ is then injective.

3.1.7. **Lemma.** *Let $L \to X$ be a line bundle. Then the element $c_1(L) \in \mathrm{H}(X)$ is nilpotent.*

Proof. By Jouanolou's trick (see [Jou73, Lemme 1.5]), there exists an affine variety X' and an affine bundle $f \colon X' \to X$. Since $f^* \colon \mathrm{H}(X) \to \mathrm{H}(X')$ is injective by (3.1.1.vii), we may replace X with X', and thus assume that X is affine. Then the line bundle L is generated by global sections, hence there exists a morphism $f \colon X \to \mathbb{P}^n$ such that

$f^*\mathcal{O}(1) \simeq L$. We may thus assume that $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$. Then by (3.1.5.b) the class $c_1(L)^s = [\mathbb{P}^{n-s}] \in \mathrm{H}(\mathbb{P}^n)$ vanishes for $s = n+1$. □

3.2. **The formal group law.**

3.2.1. **Definition.** Let R be a commutative ring. A (commutative, one-dimensional) formal group law over R is a power series $F \in R[[x,y]]$ such that:

(i) $F(x,y) = F(y,x) \in R[[x,y]]$,
(ii) $F(x,0) = x \in R[[x]]$,
(iii) $F(x, F(y,z)) = F(F(x,y),z) \in R[[x,y,z]]$.

3.2.2. **Remark.** From the axioms, we deduce that a formal group law $F \in R[[x,y]]$ must be of the form

$$F = x + y + \sum_{i,j \geqslant 1} a_{i,j} x^i y^j, \quad \text{with } a_{i,j} = a_{j,i} \in R.$$

There exists a ring $\mathbb{L}$ (the "Lazard ring") endowed with a formal group law $U \in \mathbb{L}[[x,y]]$, which is universal in the following sense: for every formal group law F over a ring R, there exists a unique ring morphism $f\colon \mathbb{L} \to R$ mapping the power series $U \in \mathbb{L}[[x,y]]$ to $F \in R[[x,y]]$.

3.2.3. **Theorem.** *There exists a power series*

$$F_{\mathrm{H}}(x,y) = \sum_{i,j \in \mathbb{N}} a_{i,j} x^i y^j \in \mathrm{H}(\operatorname{Spec} k)[[x,y]]$$

with $a_{i,j} \in \mathrm{H}^{1-i-j}(\operatorname{Spec} k)$ *such that for any line bundles* L, M *over* $X \in \mathrm{Sm}_k$

$$c_1^{\mathrm{H}}(L \otimes M) = F_{\mathrm{H}}(c_1^{\mathrm{H}}(L), c_1^{\mathrm{H}}(M)) \in \mathrm{H}(X). \tag{3.2.3.a}$$

Moreover the power series F_{H} *is a formal group law over* $\mathrm{H}(\operatorname{Spec} k)$.

Proof (sketch). Let $m, n \in \mathbb{N}$. By the axiom (3.1.1.vi), the $\mathrm{H}(\operatorname{Spec} k)$-module $\mathrm{H}(\mathbb{P}^m \times \mathbb{P}^n)$ is freely generated by the classes $c_1(\mathcal{O}(1,0))^i c_1(\mathcal{O}(0,1))^j$ for $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Therefore there are unique elements $a_{i,j}^{m,n} \in \mathrm{H}^{1-i-j}(\operatorname{Spec} k)$ such that

$$c_1(\mathcal{O}(1,1)) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j}^{m,n} c_1(\mathcal{O}(1,0))^i c_1(\mathcal{O}(0,1))^j \tag{3.2.3.b}$$

Next, one checks that $a_{i,j} = a_{i,j}^{m,n}$ does not depend on m, n as long as $i \leqslant m$ and $j \leqslant n$, and that the resulting power series

$$F_{\mathrm{H}}(x,y) = \sum_{i,j} a_{i,j} x^i y^j.$$

is actually a formal group law (details omitted, see [LM07, Corollary 4.1.8]).

Now let L, M be line bundles over $X \in \mathrm{Sm}_k$, and let us show the formula (3.2.3.a). By Jouanolou's trick (see [Jou73, Lemme 1.5]), we may assume that X is affine. Then the line bundles L, M are generated by global sections, and thus there exists a morphism $f\colon X \to \mathbb{P}^m \times \mathbb{P}^n$ for some m, n such that $f^*\mathcal{O}(1,0) \simeq L$ and $f^*\mathcal{O}(0,1) \simeq M$. We conclude by applying f^* to the equation (3.2.3.b). □

3.2.4. **Definition.** The power series F_{H} of Theorem 3.2.3 will be called the formal group law of the oriented cohomology theory H.

3.2.5. **Example.** We have $F_{\mathrm{CH}}(x,y) = x+y$ and $F_K(x,y) = x+y-txy$.

3.2.6. **Lemma.** *Let E be a vector bundle over $X \in \mathrm{Sm}_k$. Assume that E admits a filtration by subbundles whose successive quotients are line bundles $L_1,\dots,L_r$. Then for any $i \in \mathbb{N}$*

$$c_i(E) = \sigma_i(c_1(L_1),\dots,c_1(L_r)),$$

where σ_i denotes the i-th elementary symmetric polynomial in r variables.

Proof. Let $p\colon \mathbb{P}(E^\vee) \to X$ be the projective bundle. In view of the definition of the Chern classes, it will suffice to prove that

$$\prod_{i=1}^{r}(c_1(\mathcal{O}(1)) - p^*c_1(L_i)) = 0 \in \mathrm{H}(\mathbb{P}(E^\vee)).$$

We proceed by induction on the rank r of E. Using Theorem 3.2.3, we have

$$c_1(\mathcal{O}(1)) = c_1(p^*L_r \otimes p^*L_r^\vee(1)) = p^*c_1(L_r) + c_1(p^*L_r^\vee(1))g \in \mathrm{H}(\mathbb{P}(E^\vee)),$$

where

$$g = \sum_{i\geqslant 0, j\geqslant 1} a_{i,j}c_1(p^*L_r)^i c_1(p^*L_r^\vee(1))^{j-1} \in \mathrm{H}(\mathbb{P}(E^\vee)).$$

Let $E' = E/L_r$. Observe that the closed immersion $j\colon \mathbb{P}(E'^\vee) \to \mathbb{P}(E^\vee)$ is the zero-locus of a section of the line bundle $p^*L_r^\vee(1)$ transverse to the zero-section (in the sense of (3.1.1.v)), so that $c_1(p^*L_r^\vee(1)) = [\mathbb{P}(E'^\vee)] \in \mathrm{H}(\mathbb{P}(E^\vee))$. Thus $c_1(\mathcal{O}(1)) - p^*c_1(L_r) = [\mathbb{P}(E'^\vee)]\cdot g$, and so

$$\prod_{i=1}^{r}(c_1(\mathcal{O}(1)) - p^*c_1(L_i)) = j_*\Big(j^*g \cdot \prod_{i=1}^{r-1}(c_1(\mathcal{O}(1)) - p'^*c_1(L_i))\Big),$$

where $p'\colon \mathbb{P}(E'^\vee) \to X$ is the projective bundle, which vanishes by induction. □

3.2.7. **Remark.** From Lemma 3.2.6, one easily deduces the Whitney sum formula expressing the behaviour of Chern classes with respect to exact sequences of vector bundles.

3.3. **Constructing new theories.** In this section H is an oriented cohomology theory. When R is a $\mathbb{Z}$-graded ring, we denote by $R[\mathbf{b}]$ the polynomial ring over R in the variables b_i for $i \in \mathbb{N}\smallsetminus\{0\}$. The ring $R[\mathbf{b}]$ is $\mathbb{Z}$-graded by letting b_i have degree $-i$.

Consider the power series (where $b_0 = 1$)

$$\pi(x) = \sum_{i\in\mathbb{N}} b_i x^i \in \mathbb{Z}[\mathbf{b}][[x]].$$

3.3.1. **Proposition.** *There is a unique way to define for every vector bundle $E \to X$ in Sm_k an invertible element $P^{\mathrm{H}}(E) \in \mathrm{H}(X)[\mathbf{b}]^\times$ in such way that:*

(i) *$f^*P^{\mathrm{H}}(E) = P^{\mathrm{H}}(f^*E)$ for any morphism $f\colon Y \to X$ in Sm_k and vector bundle $E \to X$,*
(ii) *$P^{\mathrm{H}}(L) = \pi(c_1(L))$ when L is a line bundle over $X \in \mathrm{Sm}_k$,*
(iii) *if $0 \to E_1 \to E_2 \to E_3 \to 0$ is an exact sequence of vector bundles over $X \in \mathrm{Sm}_k$, then $P^{\mathrm{H}}(E_2) = P^{\mathrm{H}}(E_1)\cdot P^{\mathrm{H}}(E_3)$.*

Proof. Uniqueness follows from the splitting principle. Next note that, if L is a line bundle over $X \in \mathrm{Sm}_k$, then $\pi(c_1(L)) \in \mathrm{H}(X)[\mathbf{b}]$ by Lemma 3.1.7. Let E be a vector bundle over $X \in \mathrm{Sm}_k$, and let us construct the element $P^{\mathrm{H}}(E)$. We may assume that E has constant rank r. By the splitting principle, we may find a composite of projective bundles $q\colon P \to X$ such that q^*E admits a filtration by subbundles whose successive quotients are line bundles $L_1, \dots, L_r$. Then $q^*\colon \mathrm{H}(X) \to \mathrm{H}(P)$ is injective. The element

$$\prod_{i=1}^{r} \pi(c_1(q^*L_i)) \in \mathrm{H}(P)[\mathbf{b}] \tag{3.3.1.a}$$

is a symmetric polynomial in the variables $c_1(L_1), \dots, c_1(L_r)$, hence a $\mathbb{Z}$-linear combination of elementary symmetric polynomials in these variables. By Lemma 3.2.6 the element (3.3.1.a) is a polynomial in the Chern classes of q^*E, hence is the image of a unique element $P^{\mathrm{H}}(E)$ under the pullback $q^*\colon \mathrm{H}(X)[\mathbf{b}] \to \mathrm{H}(P)[\mathbf{b}]$. It is easy to verify that the element $P^{\mathrm{H}}(E)$ depends neither on the choice of the filtration, nor on the choice of $q\colon P \to X$ (the key point being that the base-change of a projective bundle remains one), and that the conditions (i) and (ii) are verified.

Moreover observe that the element (3.3.1.a) is invertible in $\mathrm{H}(P)[\mathbf{b}]$ (because $\pi(0) = 1$), and that its inverse is again a symmetric polynomial in $c_1(L_1), \dots, c_1(L_r)$, and so as above the inverse is the pullback of a unique element of $\mathrm{H}(X)[\mathbf{b}]$, which yields an inverse of $P^{\mathrm{H}}(E)$ in $\mathrm{H}(X)[\mathbf{b}]$, because q^* is injective.

Finally to prove (iii), we find a composite of projective bundles $q\colon P \to X$ such that q^*E_1 and q^*E_3 admit filtrations by subbundles whose successive quotients are line bundles (see (3.1.6)). This induces a filtration of q^*E_2 where quotients are line bundles, and it is easy to verify that $P^{\mathrm{H}}(E_2) = P^{\mathrm{H}}(E_1) \cdot P^{\mathrm{H}}(E_3)$. □

3.3.2. **Definition.** The assignment P^{H} of Proposition 3.3.1 extends uniquely to a group morphism

$$P^{\mathrm{H}}\colon K_0(X) \to \mathrm{H}(X)[\mathbf{b}]^{\times},$$

where $K_0(X)$ denotes the Grothendieck group of vector bundles over $X \in \mathrm{Sm}_k$.

For $X \in \mathrm{Sm}_k$ we set $\underline{\mathrm{H}}(X) = \mathrm{H}(X)[\mathbf{b}]$ and for a morphism $f\colon Y \to X$ in Sm_k we let $f^*_{\underline{\mathrm{H}}}\colon \underline{\mathrm{H}}(X) \to \underline{\mathrm{H}}(Y)$ be the map induced by f^*_{H}. If f is projective, for any $a \in \underline{\mathrm{H}}(Y)$ we set

$$f_*^{\underline{\mathrm{H}}}(a) = f_*^{\mathrm{H}}(P^{\mathrm{H}}(f^*T_X - T_Y)a) \in \underline{\mathrm{H}}(X).$$

3.3.3. **Proposition.** *The functor $\underline{\mathrm{H}}$, together with the above defined pushforwards, is an oriented cohomology theory.*

Proof. See [LM07, §7.4.2] or [Mer02, Proposition 4.3]. □

The formal group law of $\underline{\mathrm{H}}$ can be derived from that of H as follows. Consider the power series (where $b_0 = 1$)

$$\exp(x) = x\pi(x) = \sum_{i \in \mathbb{N}} b_i x^{i+1} \in \mathbb{Z}[\mathbf{b}][[x]].$$

Observe that, if L is a line bundle over $X \in \mathrm{Sm}_k$, then

$$c_1^{\underline{\mathrm{H}}}(L) = \exp(c_1^{\mathrm{H}}(L)) \in \underline{\mathrm{H}}(X).$$

Denoting by log the composition inverse of exp, we thus have in $\underline{\mathrm{H}}(\operatorname{Spec} k)[[x, y]]$

$$F_{\underline{\mathrm{H}}}(x, y) = \exp(F_{\mathrm{H}}(\log(x), \log(y))).$$

Taking H = CH, this yields a ring morphism

(3.3.3.a) $$\mathbb{L} \to \underline{\mathrm{CH}}(\operatorname{Spec} k) = \mathbb{Z}[\mathbf{b}].$$

which classifies the formal group law $\exp(\log x + \log y)$. Purely algebraic considerations can be used to show that the morphism (3.3.3.a) is injective (see [Ada74, II, §7]).

3.4. **The cohomology of the point.** In this section, we will describe the image of the morphism (3.3.3.a).

3.4.1. **Definition.** Let H is oriented cohomology theory. When X is a smooth projective variety, we will denote by $[\![X]\!]_{\mathrm{H}} \in \mathrm{H}(\operatorname{Spec} k)$ the element $p_*^{\mathrm{H}}(1)$, where $p\colon X \to \operatorname{Spec} k$ is the structural morphism. In addition, we will denote by $\mathrm{H}_f \subset \mathrm{H}(\operatorname{Spec} k)$ the subgroup generated by the classes $[\![X]\!]_{\mathrm{H}}$, where X runs over the smooth projective varieties.

Note that H_f is a graded subring of $\mathrm{H}(\operatorname{Spec} k)$.

3.4.2. **Example.** We have $\mathrm{CH}_f = \mathbb{Z}$ and $K_f = \mathbb{Z}[t] \subset \mathbb{Z}[t, t^{-1}]$.

The inclusion $\underline{\mathrm{CH}}_f \subset \underline{\mathrm{CH}}(\operatorname{Spec} k) = \mathbb{Z}[\mathbf{b}]$ is not surjective (as we will see later, this is related to the fact already observed that certain Chern numbers are always divisible by certain integers), and moreover admits no retraction. The situation improves somewhat if we consider K-theory instead of Chow groups:

3.4.3. **Theorem** (Hattori–Stong)**.** *The morphism of graded groups*
$$\underline{K}_f \subset \underline{K}(\operatorname{Spec} k) = \mathbb{Z}[t, t^{-1}][\mathbf{b}]$$
admits a retraction.

Proof. See [Mer02, Proposition 7.16 (1)]. In brief, there are two steps:

(i) Observe that the subring $\underline{K}_f \subset \mathbb{Z}[t, t^{-1}][\mathbf{b}]$ is contained in $\mathbb{Z}[t][\mathbf{b}]$. Then using the Riemann–Roch theorem, show that the morphism $\mathbb{Z}[t][\mathbf{b}] \to \mathbb{Z}[\mathbf{b}]$ given by $t \mapsto 0$ induces an isomorphism $\underline{K}_f \xrightarrow{\sim} \underline{\mathrm{CH}}_f$. Since $\underline{\mathrm{CH}}_f \subset \mathbb{Z}[\mathbf{b}]$, this implies that for each integer d the homogeneous component $\underline{K}_f^{-d}$ is a free abelian group of rank bounded by the number of monomials of degree $-d$ in $\mathbb{Z}[\mathbf{b}]$.

(ii) For each prime number p and integer $n \in \mathbb{N} \smallsetminus \{0\}$, construct an n-dimensional smooth projective variety M_n^p, in such a way that the classes of $[\![M_{\alpha_1}^p \times \cdots \times M_{\alpha_m}^p]\!]_{\underline{K}}$, where $b_{\alpha_1} \cdots b_{\alpha_m}$ runs over the monomials of degree $-d$, have $\mathbb{F}_p$-linearly independent images in $\mathbb{F}_p[t, t^{-1}][\mathbf{b}]$, for each $d \in \mathbb{N} \smallsetminus \{0\}$.

Combining the two steps shows that for each prime p the morphism $\underline{K}_f/p \to \mathbb{F}_p[t, t^{-1}][\mathbf{b}]$ is injective, which implies the theorem. □

3.4.4. **Remark.** The retraction is not a ring morphism!

3.4.5. **Example.** (Milnor hypersurfaces.) Let $0 \leqslant m \leqslant n$ be integers. Denote by U_m the kernel of the canonical epimorphism $1^{\oplus m+1} \to \mathcal{O}(1)$ of vector bundles over $\mathbb{P}^m$, and set $H_{m,n} = \mathbb{P}_{\mathbb{P}^m}(U_m \oplus 1^{\oplus n-m})$. Then $H_{m,n}$ is a smooth hypersurface in $\mathbb{P}^m \times \mathbb{P}^n$ defined by the vanishing of a section of the line bundle $\mathcal{O}(1,1)$ which is transverse to the zero-section (in the sense of (3.1.1.v)).

3.4.6. **Lemma.** *Let* H *be an oriented cohomology theory. The coefficients* $a_{m,n} \in \mathrm{H}(\operatorname{Spec} k)$ *of the formal group law* F_{H} *belong to the subring* H_f*.*

Proof. We proceed by induction on $m+n$. By symmetry we may assume that $m \leqslant n$. Consider the Milnor hypersurface $H_{m,n} \subset \mathbb{P}^m \times \mathbb{P}^n$. Then by (3.1.5.a), (3.1.5.b) and (3.1.5.b)

$$[H_{m,n}] = c_1(\mathcal{O}(1,1)) = \sum_{0 \leqslant i+j \leqslant m+n} a_{i,j}[\mathbb{P}^{m-i} \times \mathbb{P}^{n-j}] \in \mathrm{H}(\mathbb{P}^m \times \mathbb{P}^n). \tag{3.4.6.a}$$

Pushing forward along $\mathbb{P}^m \to \mathbb{P}^n$ yields in $\mathrm{H}(\operatorname{Spec} k)$

$$a_{m,n} = [\![H_{m,n}]\!]_{\mathrm{H}} - \sum_{0 \leqslant i+j \leqslant m+n-1} a_{i,j} \cdot [\![\mathbb{P}^{m-i} \times \mathbb{P}^{n-j}]\!]_{\mathrm{H}}$$

and we conclude by induction. □

3.4.7. **Remark.** The proof of Lemma 3.4.6 in fact shows that the coefficients $a_{m,n}$ are polynomials in the classes of Milnor hypersurfaces and projective spaces.

It follows from Lemma 3.4.6 that the morphism $\mathbb{L} \to \mathrm{H}(\operatorname{Spec} k)$ classifying the formal group law of H has image contained in the subring H_f. One may further prove:

3.4.8. **Theorem.** *If* $\mathrm{H} = \mathrm{CH}$ *or* $\mathrm{H} = K$*, then the ring morphism* $\mathbb{L} \to \underline{\mathrm{H}}_f$ *is bijective.*

Proof. See [Mer02, Proposition 6.2 (2), Theorem 8.2]. Roughly speaking the steps are the following:

(i) As already mentioned, the morphism $\mathbb{L} \to \underline{\mathrm{CH}}_f$ is injective.
(ii) Using the Riemann–Roch theorem, one shows that there exists an isomorphism of $\mathbb{L}$-algebras $\underline{K}_f \simeq \underline{\mathrm{CH}}_f$, compatibly with the classes of smooth projective varieties (Lemma 3.5.8 below).
(iii) From the proof of Theorem 3.4.3, we see that the classes $[\![M_n^p]\!]_{\underline{K}}$, where p runs over the prime numbers and n over $\mathbb{N} \smallsetminus \{0\}$, generate the ring $\underline{K}_f$.
(iv) Each variety M_n^p is a hypersurface in a product of projective spaces, hence its class in $[\![M_n^p]\!]_{\underline{K}} \in \underline{K}_f$ belongs to the $\mathbb{L}$-algebra generated by classes of projective spaces (arguing as in Lemma 3.4.6).
(v) Explicit computations show that the classes of projective spaces $[\![\mathbb{P}^n]\!]_{\underline{\mathrm{CH}}} \in \underline{\mathrm{CH}}_f \subset \mathbb{Z}[\mathbf{b}]$ are the coefficients of the derivative of the power series $\log x$ (Miščenko's formula), from which one deduces that they belong to the subring $\mathbb{L}$. □

In particular, for $\mathrm{H} \in \{\mathrm{CH}, K\}$ the ring $\underline{\mathrm{H}}_f$ does not depend on the base field k.

3.4.9. **Remark.** When k has characteristic zero, there exists an oriented cohomology theory Ω (see [LM07]) with the property that

$$\mathbb{L} \xrightarrow{\sim} \Omega_f = \Omega(\operatorname{Spec} k).$$

3.5. **Chern numbers and the Lazard ring.**

3.5.1. **Definition.** A sequence of integers $\alpha = (\alpha_1, \dots, \alpha_m)$ with $m \in \mathbb{N}$ is called a *partition* if $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m > 0$. We will write $|\alpha| = \alpha_1 + \cdots + \alpha_m \in \mathbb{N}$. To the partition α corresponds the monomial $b_\alpha = b_{\alpha_1} \cdots b_{\alpha_m} \in \mathbb{Z}[\mathbf{b}]$.

3.5.2. **Definition.** Let $X \in \mathrm{Sm}_k$ and $E \in K_0(X)$. Observe that $P^{\mathrm{H}}(E)$ has degree zero in the $\mathbb{Z}$-graded ring $\mathrm{H}(X)[\mathbf{b}]$. For each partition α, we define the *Conner–Floyd Chern class* $c_\alpha^{\mathrm{H}}(E) \in \mathrm{H}^{|\alpha|}(X)$ (or simply $c_\alpha(E)$) as the b_α-coefficient of $P^{\mathrm{H}}(E)$, so that

$$P^{\mathrm{H}}(E) = \sum_\alpha c_\alpha^{\mathrm{H}}(E) b_\alpha \in \mathrm{H}(X)[\mathbf{b}].$$

3.5.3. **Example.** When α is the partition $(1,\dots,1)$ of length n, we have $b_\alpha = b_1^n$ and $c_\alpha(E) = c_n(E)$ for any vector bundle $E \to X$ in Sm_k.

3.5.4. **Lemma.** *There are polynomials $Q_\alpha, R_\alpha \in \mathbb{Z}[\mathbf{c}]$, indexed by the partitions α, such that, for any vector bundle $E \to X$ in Sm_k*

$$c_\alpha(E) = Q_\alpha(c_1(E),\dots) \quad \textit{and} \quad c_\alpha(-E) = R_\alpha(c_1(E),\dots),$$

and every polynomial in $\mathbb{Z}[\mathbf{c}]$ is a $\mathbb{Z}$-linear combination of the polynomials Q_α, resp. R_α.

Proof (sketch). Let $\alpha = (\alpha_1,\dots,\alpha_m)$ be a partition, and let $n \geqslant m$. The n-th symmetric group acts on the ring $\mathbb{Z}[x_1,\dots,x_n]$ by permuting the variables. The sum of the elements in the orbit of $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$ may be written as a polynomial Q_α in the elementary symmetric polynomials $\sigma_1,\dots,\sigma_m$, which does not depend on the choice of n. The formula $c_\alpha(E) = Q_\alpha(c_1(E),\dots)$ then follows from the splitting principle.

If $P \in \mathbb{Z}[c_1,\dots,c_n]$, then $P(\sigma_1,\dots,\sigma_n) \in \mathbb{Z}[x_1,\dots,x_n]$ is a $\mathbb{Z}$-linear combination of orbits of monomials in $x_1,\dots,x_n$, hence P is a $\mathbb{Z}$-linear combination of polynomials Q_α. The relation $P^{\mathrm{H}}(E)P^{\mathrm{H}}(-E) = 1$ permits to express $c_\alpha(-E)$, resp. $c_\alpha(E)$, as a $\mathbb{Z}$-linear combination of $c_\beta(E)$, resp. $c_\beta(-E)$ (whose coefficients depend only on α). ☐

For any partition α, taking the b_α-coefficient yields a group morphism

$$c_\alpha \colon \mathbb{L} \subset \mathbb{Z}[\mathbf{b}] \to \mathbb{Z}.$$

3.5.5. **Definition.** When X is a smooth projective variety we will denote by $[\![X]\!] \in \mathbb{L}$ the element corresponding to $[\![X]\!]_{\underline{\mathrm{CH}}}$ under the isomorphism $\mathbb{L} \xrightarrow{\sim} \underline{\mathrm{CH}}_f$.

3.5.6. **Remark.** The group morphism c_α is given by

$$c_\alpha([\![X]\!]) = \deg\left(c_\alpha^{\mathrm{CH}}(-T_X)\right) \in \mathbb{Z},$$

for every smooth projective variety X. Therefore, in view of Lemma 3.5.4, the class $[\![X]\!] \in \mathbb{L}$ determines, and is determined by, the collection of Chern numbers of X. This yields an alternative definition of the Lazard ring: declare two smooth projective varieties equivalent if they have the same collection of Chern numbers (indexed by tuples of integers in $\mathbb{N}$). The set of equivalence classes is a commutative monoid for the disjoint union; let L be the associated abelian group. Then the cartesian product of varieties induces a ring structure on L, and we have a ring isomorphism $L \xrightarrow{\sim} \mathbb{L}$ mapping the class of a smooth projective variety X to the class $[\![X]\!]$.

3.5.7. **Remark.** Each of the mappings $X \mapsto \chi(X)$ and $X \mapsto \chi(X, \mathcal{O}_X)$ defines a ring morphism $\mathbb{L} \to \mathbb{Z}$.

3.5.8. **Lemma.** *Let X be a smooth projective variety. Then the morphism $\mathbb{L} \to \underline{K}_f$ sends $[\![X]\!]$ to $[\![X]\!]_{\underline{K}}$.*

Proof. There exists an oriented cohomology theory CK ("connective K-theory", see [Cai08] where we set $CK = \bigoplus_{d\in\mathbb{Z}} CK_{d,-d}$) such that $CK(\operatorname{Spec} k) = \mathbb{Z}[\beta]$, together with morphisms of oriented cohomology theories $CK \to \mathrm{CH}$ (such that $\beta \mapsto 0$) and $CK \to K$ (such that $\beta \mapsto t$). We thus have morphisms of $\mathbb{L}$-algebras $\underline{\mathrm{CH}}_f \leftarrow \underline{CK}_f \to \underline{K}_f$, such that $[\![X]\!]_{\underline{\mathrm{CH}}} \leftarrow\!\shortmid [\![X]\!]_{\underline{CK}} \mapsto [\![X]\!]_{\underline{K}}$. The morphism $\underline{CK}_f \to \underline{K}_f$ is bijective: it is tautologically surjective, and it is the restriction of the injective morphism $\underline{CK}(\operatorname{Spec} k) = \mathbb{Z}[\beta][\mathbf{b}] \to \underline{K}(\operatorname{Spec} k) = \mathbb{Z}[t,t^{-1}][\mathbf{b}]$. The statement follows.

Alternatively, one may prove the statement without introducing the theory CK, using instead the Riemann–Roch theorem (see [Mer02, Proposition 6.2] for details). ☐

3.5.9. **Proposition.** *Let $n \in \mathbb{Z}$, and X a smooth projective variety. Then $[\![X]\!] \in n\mathbb{L}$ if and only if*

$$\chi(X, \Lambda^{i_1} T_X \otimes \cdots \otimes \Lambda^{i_m} T_X) \in n\mathbb{Z} \quad \textit{for all } m \in \mathbb{N} \textit{ and } (i_1, \ldots, i_n) \in \mathbb{N}^m.$$

Proof. It follows from the Hattori–Stong Theorem 3.4.3 and Lemma 3.5.8 that $[\![X]\!] \in n\mathbb{L}$ if and only $p_*^K(c_\alpha^K(-T_X)) \in n\mathbb{Z}$ for all partitions α. By Lemma 3.5.4, this happens if and only if

$$p_*^K(Q(c_1^K(T_X), \ldots)) \in n\mathbb{Z}$$

for all polynomials $Q \in \mathbb{Z}[\mathbf{c}]$. We conclude with the lemma below. □

3.5.10. **Lemma.** *Let $n, r, d \in \mathbb{N}$. There are polynomials P, R in $\mathbb{Z}[t, t^{-1}][x_1, \ldots, x_r]$ such that for any rank r vector bundle $E \to X$ with $X \in \mathrm{Sm}_k$ and $\dim X \leqslant d$, we have in $K(X)$*

$$c_n(E) = P([\Lambda^1 E], \ldots, [\Lambda^r E]) \quad \textit{and} \quad [\Lambda^n E] = R(c_1(E), \ldots, c_r(E)).$$

Proof. See e.g. [Hau22, Lemma 2.2.12]. The key points are that, for vector bundles E, F over $X \in \mathrm{Sm}_k$, we have

$$c_m(E \oplus F) = \sum_{i+j=m} c_i(E) c_j(F) \quad \text{and} \quad \Lambda^m(E \oplus F) \simeq \bigoplus_{i+j=m} \Lambda^i(E) \otimes \Lambda^j(E),$$

while, when $L \to X$ is a line bundle,

$$c_1(L^\vee) = t^{-1}(1 - [L]) \quad \text{and} \quad \Lambda^1(L) = L.$$

In addition, by the existence of the formal group law, the datum of $c_1(L^\vee)$ is equivalent to that of $c_1(L)$. □

We conclude this section by supplying some information on the structure of the ring $\mathbb{L}$. For $d \in \mathbb{N} \smallsetminus \{0\}$, let us define the following integer:

$$\omega_d = \begin{cases} 1 & \text{if } d+1 \text{ is not a prime power,} \\ p & \text{if } d+1 \text{ is a power of the prime } p. \end{cases}$$

An elementary computation shows that

$$\omega_d = \gcd_{1 \leqslant i \leqslant [(d+1)/2]} \binom{d+1}{i}. \tag{3.5.10.a}$$

3.5.11. **Proposition.** *The ring $\mathbb{L}$ is polynomial on generators $y_d \in \mathbb{L}^{-d}$ for $d \in \mathbb{N} \smallsetminus \{0\}$ satisfying $c_{(d)}(y_d) = \omega_d$.*

Proof. This can be deduced from purely algebraic considerations on the map $\mathbb{L} \to \mathbb{Z}[\mathbf{b}]$ of (3.3.3.a); see [Ada74, II, §7]. □

Conversely, it is not difficult to see that a family $\ell_d \in \mathbb{L}^{-d}$ for $d \in \mathbb{N} \smallsetminus \{0\}$ constitutes a set of polynomial generators of the ring $\mathbb{L}$ if and only if $c_{(d)}(\ell_d) = \pm\omega_d$ for each $d \in \mathbb{N} \smallsetminus \{0\}$.

Explicit generators of $\mathbb{L}$ can be constructed using the classes of projective spaces and Milnor hypersurfaces. This follows from (3.5.10.a), since

$$c_{(d)}([\![\mathbb{P}^d]\!]) = -d - 1 = -\binom{d+1}{1}$$

and when $2 \leqslant m \leqslant n$, setting $d = m+n-1$,

$$c_{(d)}([\![H_{m,n}]\!]) = \binom{d+1}{m}.$$

4. Epilogue

We are now in position to reformulate Theorem 2.2.5 in terms of cobordism:

4.1. **Theorem.** *Let X be a smooth projective variety with an action of a p-group G. Assume that one of the following conditions holds:*

(i) *G is cyclic.*
(ii) $\operatorname{char} k = p$.
(iii) $\dim X < p-1$.

If $[\![X]\!] \notin p\mathbb{L}$, then $X^G \neq \varnothing$.

Proof. In view of Proposition 3.5.9, this follows by applying Theorem 2.2.5 to the G-equivariant $\mathcal{O}_X$-modules $\mathcal{F} = \Lambda^{\alpha_1} T_X \otimes \cdots \otimes \Lambda^{\alpha_m} T_X$. □

4.2. **Definition.** For $n \in \mathbb{N} \smallsetminus \{0\}$, we denote by $I_p(n)$ the ideal of $\mathbb{L}$ generated by the classes $[\![X]\!]$, where X runs over the smooth projective varieties of dimension $\leqslant p^{n-1}-1$ having all Chern numbers divisible by p. So $I_p(1) = p\mathbb{L}$, and $I_p(n) \subset I_p(n+1)$. We also set $I_p(0) = 0 \subset \mathbb{L}$, and $I_p(\infty) = \bigcup_n I_p(n)$.

The next statement bridges the gap between (2.2.5.i) (case $r = 1$) and (2.1.2.i) (using the inclusion $I_p(r) \subset I_p(\infty)$).

4.3. **Theorem** (Work in progress)**.** *Let G be an abelian p-group of rank r, and X a smooth projective variety with an action of G. If $[\![X]\!] \notin I_p(r)$, then $X^G \neq \varnothing$.*

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