Intersection Theory

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Contents

CHAPTER 1

Algebraic cycles

Basic references are [**Ful98**], [**EKM08**, Chapters IX and X] and [**Sta18**, Tag 02P3].

1. Length of a module

All rings will be commutative, with unit, and noetherian. When A is a local ring, we denote by $\mathfrak{m}_A$ its maximal ideal.

Let A be a (noetherian commutative) ring, and M a finitely generated A-module. The length of a chain of submodules $0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ is the integer n. The *length of* M, denoted by

$$l_A(M) \in \mathbb{N} \cup \{\infty\}$$

is supremum of the length of the chains of submodules of M. If I is an ideal of M such that $IM = 0$, then $l_A(M) = l_{A/I}(M)$. When A is a field, then $l_A(M)$ is the dimension of the A-vector space M. The length of the ring A is $l_A(A)$ and will be denoted by $l(A)$.

Definition 1.1.1. A function ψ, which associates to every finitely generated A-module M an element $\psi(M)$ of $\mathbb{N}\cup\{\infty\}$ will be called *additive*, if for every exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0$$

we have in $\mathbb{N} \cup \{\infty\}$,

$$\psi(M) = \psi(M') + \psi(M'').$$

Proposition 1.1.2. *The length function $M \mapsto l_A(M)$ is additive.*

The support of M, denoted $\operatorname{Supp} M$, is the set of primes $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$. The dimension of M, denoted $\dim M$, is the Krull dimension of the topological space $\operatorname{Supp} M$. It coincides with the dimension of the ring $A/\operatorname{Ann}(M)$.

Proposition 1.1.3. *Let A be a local ring, and M a finitely generated A-module. The following conditions are equivalent:*

(i) $l_A(M) < \infty$.
(ii) There is $n \in \mathbb{N}$ *such that* $(\mathfrak{m}_A)^n M = 0$.
(iii) We have $\dim M \leq 0$.

Lemma 1.1.4. *Let M be a finitely generated A-module. There is a sequence of A-submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that*

$$M_{i+1}/M_i \simeq A/\mathfrak{p}_i$$

with $\mathfrak{p}_i \in \operatorname{Supp} M$.

2. Group of cycles

We fix a base field k. A *variety* will mean a separated scheme of finite type over $\operatorname{Spec} k$. Unless otherwise specified, all schemes will be assumed to be varieties, and a morphism will be a k-morphism. The function field of an integral variety X will be denoted by $k(X)$. If Z is an integral closed subscheme of a variety X, we denote by $\mathcal{O}_{X,Z}$ the local ring $\mathcal{O}_{X,z}$ at the generic point z of Z.

DEFINITION 1.2.1. Let X be a variety. We define $\mathcal{Z}(X)$ as the free abelian group on the classes V of integral closed subschemes V of X. A *cycle on* X is an element of $\mathcal{Z}(X)$, that is, a finite $\mathbb{Z}$-linear combination of elements $[V]$, for V as above. There is a grading $\mathcal{Z}(X) = \bigoplus_n \mathcal{Z}_n(X)$, where $\mathcal{Z}_n(X)$ is the subgroup generated by the classes $[V]$ with $\dim V = n$.

DEFINITION 1.2.2. When T is a (possibly non-integral) closed subscheme of X, we define its class

$$[T] = \sum_i m_i[T_i] \in \mathcal{Z}(X),$$

where T_i are the irreducible components of T, and $m_i = l(\mathcal{O}_{T,T_i})$ is the multiplicity of T at T_i. (The local ring $\mathcal{O}_{T,T_i}$ has dimension zero, hence finite length by Proposition 1.1.3; there are only finitely many irreducible components because T is a noetherian scheme.) Note that $[\varnothing] = 0$.

DEFINITION 1.2.3. Let $Y \to X$ be a dominant morphism between integral varieties. We define an integer

$$\deg(Y/X) = \begin{cases} [k(Y) : k(X)] & \text{if } \dim Y = \dim X, \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 1.2.4. When $f\colon Y \to X$ is a morphism (between varieties), and W an integral closed subscheme of Y, we let V be the closure of $f(W)$ in X (or equivalently the scheme-theoretic image of $W \to X$), and define

$$f_*[W] = \deg(W/V) \cdot [V]$$

This extends by linearity to give a group homomorphism

$$f_*\colon\ \mathcal{Z}_n(Y) \to \mathcal{Z}_n(X).$$

EXAMPLE 1.2.5. Let X be a variety, with structural morphism $p\colon X \to \operatorname{Spec} k$. Then we have a group homomorphism

$$\deg = p_*\colon\ \mathcal{Z}(X) \to \mathcal{Z}(\operatorname{Spec} k) = \mathbb{Z}.$$

We have $\deg \mathcal{Z}_n(X) = 0$ if $n > 0$. The group $\mathcal{Z}_0(X)$ is generated by the classes of closed points of X, and for such a point x with residue field $k(x)$, we have

$$\deg[\{x\}] = [k(x) : k].$$

LEMMA 1.2.6. *Consider morphisms* $Z \xrightarrow{g} Y \xrightarrow{f} X$. *We have*

$$(f \circ g)_* = f_* \circ g_*\colon\ \mathcal{Z}(Z) \to \mathcal{Z}(X).$$

PROOF. Let W be an integral closed subscheme of Z. Let V be the closure of $g(W)$ in Y, and U the closure of $f(V)$ in X. Then U is the closure of $(f \circ g)(W)$ in X. We have $\dim W = \dim U$ if and only if $\dim V = \dim U$ and $\dim V = \dim W$, in which case

$$\begin{aligned}(f \circ g)_*[W] &= [k(W) : k(U)] \cdot [U] \\ &= [k(W) : k(V)] \cdot [k(V) : k(U)] \cdot [U] \\ &= [k(W) : k(V)] \cdot f_*[V] \\ &= f_* \circ g_*[W].\end{aligned}$$

Otherwise $(f \circ g)_*[W] = 0$, and either $g_*[W] = 0$ or $f_*[V] = 0$. Since $f_*[V]$ is a multiple of $[W]$, we have $f_* \circ g_*[W] = 0$ in either case. □

3. Effective Cartier divisors I

DEFINITION 1.3.1. A closed embedding $D \to X$ is called an *effective Cartier divisor* if its ideal $\mathcal{I}_D$ is a locally free $\mathcal{O}_X$-module of rank one (i.e. an invertible $\mathcal{O}_X$-module). It is equivalent to require that every point of X have an open affine neighborhood $U = \operatorname{Spec} A$ such that $D \cap U = \operatorname{Spec} A/aA$ for some nonzerodivisor $a \in A$ (recall that $a \in A$ is called a nonzerodivisor if the only $x \in A$ such that $ax = 0$ is $x = 0$).

PROPOSITION 1.3.2. *Let $f\colon Y \to X$ be morphism, and $D \to X$ an effective Cartier divisor. Then $f^{-1}D \to Y$ is an effective Cartier divisor, under any of the following assumptions.*
(i) Y is integral, and $f^{-1}D \neq Y$,
(ii) or f is flat.

PROOF. We may assume that $X = \operatorname{Spec} A$, and $D = \operatorname{Spec} A/aA$ for some nonzerodivisor $a \in A$. We may further assume that $Y = \operatorname{Spec} B$, that f is given by a ring morphism $u\colon A \to B$, and prove that $u(a)$ is a nonzerodivisor in B.

If $u\colon A \to B$ is flat, then multiplication by a is an injective endomorphism of A, hence multiplication by $u(a) = a \otimes 1$ is an injective endomorphism of $B = A \otimes_A B$ (by flatness), so that $u(a)$ is a nonzerodivisor in B.

If $f^{-1}D \neq Y$, then the element $u(a) \in B$ is nonzero, hence a nonzerodivisor if B is a domain (i.e. Y is integral). □

We will use the following version of Krull's principal ideal theorem:

THEOREM 1.3.3. *Let A be a noetherian ring and $a \in A$ a nonzerodivisor. Then every prime of A minimal over a has height one.*

LEMMA 1.3.4. *Let $D \to X$ be an effective Cartier divisor, with X of pure dimension n. Then D has pure dimension $n-1$.*

PROOF. To prove that D has pure dimension $n-1$, we may assume that $X = \operatorname{Spec} A$ and $D = \operatorname{Spec} A/aA$ for some nonzerodivisor $a \in A$. Then the irreducible components of D correspond to the minimal primes of A over a. If $\mathfrak{p}$ is such a prime, then $\operatorname{height} \mathfrak{p} = 1$ by Krull's Theorem 1.3.3. Let $\mathfrak{q}$ be a minimal prime of A contained in $\mathfrak{p}$, and T the corresponding irreducible component of X. We recall that in an integral domain which is finitely generated over a field, all the maximal chains of primes have the same length (see e.g. [**Har77**, Theorem 1.8A]). In particular

$$\dim T = \operatorname{tr.deg.}(k(T)/k) = n = 1 + \dim A/\mathfrak{p},$$

so that the irreducible component of D corresponding to $\mathfrak{p}$ has dimension $n-1$. □

Proposition 1.3.5. *Let X be an equidimensional variety, and $D \to X$ an effective Cartier divisor. Let X_i be the irreducible components of X, and $m_i = l(\mathcal{O}_{X,X_i})$ the corresponding multiplicities. Then*

$$[D] = \sum_i m_i [D \cap X_i] \in \mathcal{Z}(X).$$

Proof. It will suffice to compare the coefficients at an integral closed subscheme Z of codimension one in X contained in D. Let $A = \mathcal{O}_{X,Z}$ and $U = \operatorname{Spec} B$ an open affine subscheme of X containing the generic point of Z such that $D \cap U \to U$ is defined by a nonzerodivisor $b \in B$. Let $a \in A$ be the image of b. Then $\mathcal{O}_{D,Z} = A/aA$, and the formula that we need to prove becomes

$$l(A/aA) = \sum_i l(A_{\mathfrak{p}_i}) l(A/(\mathfrak{p}_i + aA)),$$

where $\mathfrak{p}_i$ are the minimal primes of A, corresponding to the components X_i containing Z (if $Z \not\subset X_j$ then the coefficient of $[D \cap X_j]$ at Z is zero). We prove the formula above in Corollary 1.4.6 in the next section. □

4. Herbrand Quotients I

Let A be a noetherian ring and $a \in A$. Let M be a finitely generated A-module. We will denote the a-torsion submodule of M by

$$M\{a\} = \ker(M \xrightarrow{a} M) = \{m \in M | am = 0\}.$$

Lemma 1.4.1. *We have* $\operatorname{Supp}(M\{a\}) \subset \operatorname{Supp}(M/aM)$.

Proof. Let $\mathfrak{p} \in \operatorname{Supp}(M\{a\})$. Then $0 \neq (M\{a\})_{\mathfrak{p}} = M_{\mathfrak{p}}\{a\} \subset M_{\mathfrak{p}}$. If $\mathfrak{p} \notin \operatorname{Supp}(M/aM)$, then $0 = (M/aM)_{\mathfrak{p}} = M_{\mathfrak{p}}/aM_{\mathfrak{p}}$, hence by Nakayama's lemma $a \notin \mathfrak{p}$. Thus $a \in (A_{\mathfrak{p}})^{\times}$, hence multiplication by a induces an injective endomorphism of $M_{\mathfrak{p}}$, so that $M_{\mathfrak{p}}\{a\} = 0$, a contradiction. □

Definition 1.4.2. Assume that $l_A(M/aM) < \infty$. Then $l_A(M\{a\}) < \infty$ by Lemma 1.4.1, and we define the integer

$$e_A(M, a) = l_A(M/aM) - l_A(M\{a\}).$$

Lemma 1.4.3. *If M has finite length, then $e_A(M, a) = 0$.*

Proof. This follows by additivity of the length function from the exact sequences of A-modules of finite length

$$0 \to M\{a\} \to M \to aM \to 0$$
$$0 \to aM \to M \to M/aM \to 0.$$ □

The next statement asserts that the function $e_A(-, a)$ is additive:

Lemma 1.4.4. *Consider an exact sequence of finitely generated A-modules*

$$0 \to M' \to M \to M'' \to 0.$$

If M/aM has finite length, then so have M'/aM' and M''/aM'', and

$$e_A(M, a) = e_A(M', a) + e_A(M'', a).$$

Proof. The snake lemma gives an exact sequence

$$0 \to M'\{a\} \to M\{a\} \to M''\{a\} \to M'/aM' \to M/aM \to M''/aM'' \to 0.$$

If M/aM has finite length, then so has its quotient M''/aM''. By Lemma 1.4.1, the A-module $M''\{a\}$ also has finite length, hence by the sequence above so has M'/aM'. The equality follows from the additivity of the length function. □

Proposition 1.4.5. *Let A be a noetherian ring and M a finitely generated A-module. Let $a \in A$ be such that the A-module M/aM has finite length. Then*

$$e_A(M,a) = \sum_{\mathfrak{p}} l_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot l(A/(\mathfrak{p} + aA)),$$

where $\mathfrak{p}$ runs over the non-maximal primes of A.

Proof. Both sides are additive in M by Proposition 1.1.2 and Lemma 1.4.4. Thus by we may assume that $M = A/\mathfrak{q}$ for some prime $\mathfrak{q}$ of A. If $\mathfrak{q}$ is maximal, then both sides vanish, in view of Lemma 1.4.3. We may thus assume that the ideal $\mathfrak{q}$ is not maximal. Since $l(A/(\mathfrak{q} + aA)) < \infty$, every prime containing $\mathfrak{q} + aA$ is maximal, and in particular $a \notin \mathfrak{q}$. By Krull's Theorem 1.3.3, we have $\dim A/\mathfrak{q} = 1$, hence the only non-maximal prime $\mathfrak{p}$ such that $M_\mathfrak{p} \neq 0$ is $\mathfrak{p} = \mathfrak{q}$. Thus the right hand side is $l_{A_\mathfrak{q}}(\kappa(\mathfrak{q})) \cdot l(A/(\mathfrak{q} + aA)) = l(A/(\mathfrak{q} + aA))$ (where $\kappa(\mathfrak{q}) = (A/\mathfrak{q})_\mathfrak{q}$ is the residue field at $\mathfrak{q}$), and coincides with the left hand side, since $M\{a\} = 0$. □

Corollary 1.4.6. *Let A be a noetherian ring of dimension one and $a \in A$ a nonzerodivisor. Then*

$$l(A/aA) = \sum_{\mathfrak{p}} l(A_\mathfrak{p}) l(A/(\mathfrak{p} + aA)),$$

where $\mathfrak{p}$ runs over the minimal primes of A.

Proof. We have $\dim A/aA \leq 0$ by Krull's Theorem 1.3.3 (or more simply because the nonzerodivisor a cannot belong to any minimal prime), hence $l(A/aA) < \infty$. Since $A\{a\} = 0$, it follows that $e_A(A,a) = l(A/aA)$. Thus the statement follows from Proposition 1.4.5 applied with $M = A$. □

5. Flat pull-back

We will make repeated use of the following form of the going-down theorem:

Proposition 1.5.1. *Let $f : Y \to X$ be a flat morphism. Then any irreducible component of Y dominates an irreducible component of X*

Definition 1.5.2. A morphism $f : Y \to X$ is said to have *relative dimension* d, if for all morphisms $V \to X$ with V integral, the variety $f^{-1}V = V \times_X Y$ has pure dimension $d + \dim V$.

If f has relative dimension d, then the same is true for any base-change of f.

Example 1.5.3. Examples of flat morphisms of relative dimension d include:

- Open immersions ($d = 0$),
- Vector bundles of constant rank d,
- Projective bundles of constant rank $d + 1$.
- The structural morphism to $\operatorname{Spec} k$ of a variety of pure dimension d.

- More generally, any flat morphism $Y \to X$ where X is irreducible and Y has pure dimension $d + \dim X$.

DEFINITION 1.5.4. Let $f\colon Y \to X$ be a flat morphism of relative dimension d. When V is an integral closed subscheme of X, we define (using Definition 1.2.2)

$$f^*[V] = [f^{-1}V] = [V \times_X Y] \in \mathcal{Z}(Y).$$

This extends by linearity to give a group homomorphism

$$f^*\colon\ \mathcal{Z}_n(X) \to \mathcal{Z}_{n+d}(Y).$$

REMARK 1.5.5. Let $u\colon U \to X$ be an open immersion. The homomorphism $u^*\colon\ \mathcal{Z}(X) \to \mathcal{Z}(U)$ sends $[V]$ to $[V \cap U]$. Note that if U_i is a finite open cover of X, the homomorphism $\mathcal{Z}(X) \to \bigoplus_i \mathcal{Z}(U_i)$ is injective.

LEMMA 1.5.6. *Let $f\colon Y \to X$ be a flat morphism with a relative dimension. Then $f^*[X] = [Y]$ in $\mathcal{Z}(Y)$.*

PROOF. Let W be an irreducible component of Y, and V the closure of its image in X. Proposition 1.5.1 implies that V is an irreducible component of X. The coefficient of $[Y]$ at W is $l(\mathcal{O}_{Y,W})$, and the coefficient of $f^*[X]$ at W is $l(\mathcal{O}_{X,V})l(\mathcal{O}_{f^{-1}V,W})$. Let $A = \mathcal{O}_{X,V}$ and $B = \mathcal{O}_{Y,W}$. Since $B/\mathfrak{m}_A B = \mathcal{O}_{f^{-1}V,W}$, we need to prove that

$$l(B) = l(A)l(B/\mathfrak{m}_A B).$$

This follows from Lemma 1.5.7 below (with $M = A$). □

LEMMA 1.5.7. *Let A be a local ring and B a flat A-algebra. Assume that $\dim A = \dim B = 0$ and let M be a finitely generated A-module. Then*

$$l_B(M \otimes_A B) = l_A(M)l(B/\mathfrak{m}_A B).$$

PROOF. Both sides are additive in M, and we may assume by Lemma 1.1.4 that $M = A/\mathfrak{m}_A$. Then $l_A(M) = 1$, and the result follows. □

PROPOSITION 1.5.8. *If $g\colon Z \to Y$ and $f\colon Y \to X$ are two flat morphisms having a relative dimension, then so is the composite $f \circ g$, and we have*

$$(f \circ g)^* = g^* \circ f^*\colon\ \mathcal{Z}(X) \to \mathcal{Z}(Z).$$

PROOF. The first statement follows at once from the definition.

Let U be an integral closed subscheme of X, and $V = f^{-1}U$ and $W = (f \circ g)^{-1}U$. Replacing $Z \to Y \to X$ with $W \to V \to U$, it will suffice to prove that the two homomorphisms have the same effect on $[X]$. By Lemma 1.5.6, we have

$$(f \circ g)^*[X] = [(f \circ g)^{-1}X] = [g^{-1}f^{-1}X] = g^*[f^{-1}X] = g^* \circ f^*[X]. \qquad \square$$

PROPOSITION 1.5.9. *Consider a cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ Y & \xrightarrow{f} & X \end{array}$$

where the morphism x (and therefore also y) is flat of relative dimension d. Then

$$f'_* \circ y^* = x^* \circ f_*.$$

PROOF. The case when f is a closed embedding follows from the definition of the flat pull-back. We prove that the two homomorphisms have the same effect on the class on an integral closed subscheme W of Y. Let V be the closure $f(W)$ in X. Taking the base change along $V \to X$, and using the case of a closed embedding, we are reduced to assuming that Y and X are integral, and that f is dominant, and (since $y^*[Y] = [Y']$ by Lemma 1.5.6) proving that

$$f'_*[Y'] = x^* \circ f_*[Y]. \tag{1.5.a}$$

Since f has relative dimension d, for every irreducible component R of Y', we have

$$\dim R - \dim X' = \dim Y - \dim X.$$

In particular, if $\dim Y > \dim X$, then $f_*[Y] = 0$ and $f'_*[Y'] = 0$, so that (1.5.a) holds.

Thus we assume that $\dim X = \dim Y$ (recall that f is dominant). We prove that the two sides of (1.5.a) have the same coefficient on the class of a given irreducible T component of X' (which must dominate X by Proposition 1.5.1). We let $K = k(X)$, $L = k(Y)$, $C = \mathcal{O}_{X',T}$, and $D = C \otimes_K L$. Applying Lemma 1.5.10 below with $M = C$, we see that the coefficient of $x^* \circ f_*[Y]$ at $[T]$ is

$$[L : K]l(C) = l_C(D).$$

The ring D is artinian (being finite over C), and the set $\operatorname{Spec} D$ is in bijection with the irreducible components of Y' dominating T. Moreover if $\mathfrak{q} \in \operatorname{Spec} D$ corresponds to an irreducible component Q, then the local rings $D_\mathfrak{q}$ and $\mathcal{O}_{Y',Q}$ are isomorphic. It follows that the coefficient of $f'_*[Y']$ at $[T]$ is

$$\sum_{\mathfrak{q}} l(D_\mathfrak{q})[D/\mathfrak{q} : C/\mathfrak{m}_C] = \sum_{\mathfrak{q}} l(D_\mathfrak{q}) l_C(D/\mathfrak{q}),$$

where $\mathfrak{q}$ runs over $\operatorname{Spec} D$. The statement follows from Lemma 1.5.11 below, applied with $A = C$ and $B = M = D$. □

LEMMA 1.5.10. *Let L/K be a finite field extension and C a K-algebra. Let M be a C-module of finite length. Then*

$$l_C(M \otimes_K L) = [L : K] l_C(M).$$

PROOF. By Proposition 1.1.3 the set $\operatorname{Supp}_C M$ consists of maximal ideals of C. Both sides of the equation are additive in M, hence by Lemma 1.1.4 we may assume that $M = C/\mathfrak{p}$, for $\mathfrak{p}$ a maximal ideal of C. Then M is a field, so that $l_C(M) = 1$, and

$$l_C(M \otimes_K L) = l_M(M \otimes_K L) = \dim_M(M \otimes_K L) = \dim_K L = [L : K],$$

where $\dim_M$ and $\dim_K$ stand for the dimensions as vector spaces. The statement follows. □

LEMMA 1.5.11. *Let B be an A-algebra, and M a B-module. Assume that M has finite length as an A-module and that $\dim B = 0$. Then*

$$l_A(M) = \sum_{\mathfrak{q} \in \operatorname{Spec} B} l_{B_\mathfrak{q}}(M_\mathfrak{q}) l_A(B/\mathfrak{q}).$$

PROOF. Both sides of the equation are additive in the B-module M. By Lemma 1.1.4 we may assume that $M = B/\mathfrak{q}$, for $\mathfrak{q}$ a maximal ideal of B, in which case both sides of the equation are equal to 1. □

CHAPTER 2

Rational equivalence

1. Order function

Let A be a local domain of dimension one and K its fraction field. When $a \in A - \{0\}$, the ring A/aA has dimension ≤ 0, hence finite length, and we define an integer

$$\operatorname{ord}_A(a) = l(A/aA) \in \mathbb{N}.$$

If $a, b \in A - \{0\}$, we have an exact sequence of A-modules

$$0 \to aA/abA \to A/abA \to A/aA \to 0.$$

Multiplication by the nonzero element a of the domain A induces an isomorphism

$$A/bA \to aA/abA.$$

Using the additivity of the length function, we deduce from the exact sequence above that

$$\operatorname{ord}_A(ab) = \operatorname{ord}_A(a) + \operatorname{ord}_A(b).$$

This allows us to extend the function ord_A to a group homomorphism from the group of invertible elements in K

$$\operatorname{ord}_A \colon K^\times \to \mathbb{Z}.$$

Concretely, we may write any $\varphi \in K^\times$ as $\varphi = f/g$ with $f, g \in A - \{0\}$ and define

$$\operatorname{ord}_A(\varphi) = l(A/fA) - l(A/gA) \in \mathbb{Z}.$$

LEMMA 2.1.1. *Let A be a discrete valuation ring with fraction field K. Then* $\operatorname{ord}_A \colon K^\times \to \mathbb{Z}$ *is the valuation of A.*

PROOF. Let π be a uniformiser of A. Any $\varphi \in K^\times$ may be written as $\varphi = \pi^n u$ with $u \in A^\times$, and $n \in \mathbb{Z}$ the valuation of φ. Observe that $\operatorname{ord}_A(u) = 0$ because $u \in A^\times$, while $\operatorname{ord}_A(\pi) = 1$ since $A/\pi A$ is the residue field of A, an A-module of length one. Thus

$$\operatorname{ord}_A(\varphi) = n \operatorname{ord}_A(\pi) + \operatorname{ord}_A(u) = n. \qquad \square$$

Let X be an integral variety, and $\varphi \in k(X)^\times$. For any point x of codimension one in X, the local ring $\mathcal{O}_{X,x}$ has dimension one and its fraction field is $k(X)$. We will write $\operatorname{ord}_x(\varphi) = \operatorname{ord}_{\mathcal{O}_{X,x}}(\varphi)$. Similarly, for an integral closed subscheme V of codimension one in X, we write $\operatorname{ord}_V(\varphi) = \operatorname{ord}_{\mathcal{O}_{X,V}}(\varphi)$.

LEMMA 2.1.2. *Let A be a finitely generated k-algebra which is a domain, $a \in A - \{0\}$, and consider the closed subscheme $D = \operatorname{Spec} A/aA$ of $X = \operatorname{Spec} A$. Then*

$$[D] = \sum_V \operatorname{ord}_V(a) \cdot [V] \in \mathcal{Z}(X),$$

where V runs over the integral closed subschemes of codimension one in X.

PROOF. Since $D \to X$ is an effective Cartier divisor, the variety D has pure dimension $\dim X - 1$ by Lemma 1.3.4. Let V be an integral closed subscheme of codimension one in X, and let $\mathfrak{p}$ be the corresponding prime of height one in A, so that $A_{\mathfrak{p}} = \mathcal{O}_{X,V}$. We have

$$l((A/aA)_{\mathfrak{p}}) = l(A_{\mathfrak{p}}/aA_{\mathfrak{p}}) = \operatorname{ord}_V(a).$$

If $V \subset D$, then the integer above is the coefficient of $[D]$ at $[V]$. If $V \not\subset D$, then the coefficient of $[D]$ at $[V]$ vanishes. But in this case we have $a \notin \mathfrak{p}$, and thus $(A/aA)_{\mathfrak{p}} = 0$, so that $\operatorname{ord}_V(a) = 0$, as required. □

PROPOSITION 2.1.3. *Let X be an integral variety, and $\varphi \in k(X)^{\times}$. The set of integral closed subschemes V of codimension one in X such that* $\operatorname{ord}_V(\varphi) \neq 0$ *is finite.*

PROOF. Taking a finite cover by open affine subschemes, we may assume that $X = \operatorname{Spec} A$. Further, we may assume that $\varphi \in A$. Then the result follows from Lemma 2.1.2. □

DEFINITION 2.1.4. Let X be an integral variety, and $\varphi \in k(X)^{\times}$. We set

$$\operatorname{div} \varphi = \sum_V \operatorname{ord}_V(\varphi) \cdot [V] \in \mathcal{Z}(X),$$

where V runs over the integral closed subschemes of codimension one in X.

Thus Lemma 2.1.2 amounts to:

LEMMA 2.1.5. *Let A be a finitely generated k-algebra which is a domain, $a \in A - \{0\}$, and consider the closed subscheme $D = \operatorname{Spec} A/aA$ of $X = \operatorname{Spec} A$. Then*

$$[D] = \operatorname{div} a \in \mathcal{Z}(X).$$

DEFINITION 2.1.6. Let X be a variety. We let $\mathcal{R}(X)$ be the subgroup of $\mathcal{Z}(X)$ generated by the elements $\operatorname{div} \varphi \in \mathcal{Z}(V) \subset \mathcal{Z}(X)$, where V runs over the integral closed subschemes of X, and $\varphi \in k(V)^{\times}$. Then we define the Chow group of X as

$$\operatorname{CH}(X) = \mathcal{Z}(X)/\mathcal{R}(X) = \bigoplus_n \operatorname{CH}_n(X),$$

where $\operatorname{CH}_n(X) = \mathcal{Z}_n(X)/\mathcal{R}_n(X)$ with $\mathcal{R}_n(X) = \mathcal{R}(X) \cap \mathcal{Z}_n(X)$.

2. Flat pull-back

When $f \colon Y \to X$ is a dominant morphism between integral varieties, and $\varphi \in k(X)^{\times}$, we define $f^*\varphi$ as the image of φ under the natural morphism $k(X)^{\times} \to k(Y)^{\times}$.

LEMMA 2.2.1. *Let $f \colon Y \to X$ be a flat morphism having a relative dimension, and let Y_i be the irreducible components of Y, with multiplicities $m_i = \mathcal{O}_{Y,Y_i}$. Assume that X is integral, and let $\varphi \in k(X)^{\times}$. Let $f_i \colon Y_i \to Y$ be the morphisms induced by f (which are dominant by Proposition 1.5.1). Then*

$$f^* \circ \operatorname{div} \varphi = \sum_i m_i \operatorname{div}(f_i^* \varphi) \in \mathcal{Z}(Y).$$

PROOF. First observe that the statement certainly holds when f is an open immersion (if $x \in Y \subset X$, then the local rings $\mathcal{O}_{Y,x}$ and $\mathcal{O}_{X,x}$ are isomorphic).

In general, since f^* and div are both compatible with restriction to open subschemes, we may assume that $X = \operatorname{Spec} A$, and also that $Y = \operatorname{Spec} B$. Then $\varphi = a/b$ with $a, b \in A$, and we may assume that $\varphi \in A$. Then φ defines an effective Cartier divisor $D \to X$. Since f is flat, its inverse image $f^{-1}D \to Y$ remains an effective Cartier divisor by Proposition 1.3.2. By the same proposition, since $f^{-1}D$ does not contain Y_i (e.g. by Lemma 1.3.4), the closed embedding $Y_i \cap f^{-1}D \to Y_i$ is an effective Cartier divisor; it is given by the element $f_i^*\varphi \in H^0(Y_i, \mathcal{O}_{Y_i})$. Using Lemma 1.5.6, Proposition 1.3.5 and Lemma 2.1.5, we have in $\mathcal{Z}(Y)$

$$f^* \circ \operatorname{div} \varphi = f^*[D] = [f^{-1}D] = \sum_i m_i [Y_i \cap f^{-1}D] = \sum_i m_i \operatorname{div}(f_i^* \varphi). \qquad \square$$

PROPOSITION 2.2.2. *Let $f : Y \to X$ be a flat morphism of relative dimension d. Then $f^* \mathcal{R}(X) \subset \mathcal{R}(Y)$, giving a group homomorphism*

$$f^* \colon \operatorname{CH}_\bullet(X) \to \operatorname{CH}_{\bullet+d}(Y).$$

PROOF. Let V be an integral closed subscheme of X, and $\varphi \in k(V)^\times$. It will suffice to prove that $f^* \circ \operatorname{div} \varphi = 0$ in $\operatorname{CH}(f^{-1}V)$. Since the morphism $f^{-1}V \to V$ is flat of relative dimension d, we may assume that X is integral and $\varphi \in k(X)^\times$. Then the statement follows from Lemma 2.2.1. $\square$

3. Localisation sequence

Let $i \colon Y \to X$ be a closed embedding. Then $i_* \mathcal{R}(Y) \subset \mathcal{R}(X)$ by definition. This gives a group homomorphism $i_* \colon \operatorname{CH}(Y) \to \operatorname{CH}(X)$.

PROPOSITION 2.3.1 (Localisation sequence). *Let $i \colon Y \to X$ be a closed embedding, and $u \colon U = X - Y \to X$ be the open complement. Then the following sequence is exact:*

$$\operatorname{CH}(Y) \xrightarrow{i_*} \operatorname{CH}(X) \xrightarrow{u^*} \operatorname{CH}(U) \to 0$$

PROOF. The following sequence is

$$0 \to \mathcal{Z}(Y) \xrightarrow{i_*} \mathcal{Z}(X) \xrightarrow{u^*} \mathcal{Z}(U) \to 0 \tag{2.3.b}$$

is (split-)exact. Thus it will suffice to take $\alpha \in \mathcal{Z}(X)$ such that $u^*\alpha = 0$ in $\operatorname{CH}(U)$, and find $\beta \in \mathcal{Z}(Y)$ such that $\alpha = i_*\beta$ in $\operatorname{CH}(X)$. There are finitely many integral closed subschemes V_j of U, and rational functions $\varphi_j \in k(V_j)^\times$ such that

$$u^*\alpha = \sum_j \operatorname{div} \varphi_j \in \mathcal{Z}(U).$$

For each j, let $\overline{V_j}$ be the closure V_j in X, and ψ_j the rational function on $\overline{V_j}$ corresponding to φ_j under the isomorphism $k(V_j) \simeq k(\overline{V_j})$. Then

$$u^*(\alpha - \sum_j \operatorname{div} \psi_j) = 0 \in \mathcal{Z}(U).$$

Using the sequence (2.3.b), we find an element $\beta \in \mathcal{Z}(Y)$ such that

$$\alpha - \sum_j \operatorname{div} \psi_j = i_*\beta \in \mathcal{Z}(X).$$

It follows that $\alpha = i_*\beta$ in $\operatorname{CH}(X)$. $\square$

CHAPTER 3

Proper push-forward

1. Distance between lattices

Let R be a local (commutative noetherian) domain of dimension one, and K its fraction field. Let V be a K-vector space of finite dimension. A *lattice in* V is a finitely generated R-submodule M of V such that the induced morphism $M \otimes_R K \to V$ is surjective (it is always injective). This means that M contains a K-basis of V.

EXAMPLE 3.1.1. Let $R \to S$ be a finite injective ring morphism. Assume that S is a domain, with fraction field L. Then the K-vector space L is finite dimensional, and S is a lattice in L. Indeed the K-algebra $S \otimes_R K$ is contained in L, hence it has finite dimension as a K-vector space and is a domain. Thus $S \otimes_R K$ is a field, and we conclude that $S \otimes_R K = L$.

LEMMA 3.1.2. *(i) If a finitely generated R-submodule M of V contains a lattice N in V, then M is a lattice in V.*

(ii) If M is a lattice in V, and φ a K-automorphism of V, then $\varphi(M)$ is a lattice in V.

PROOF. (i) : Indeed the morphism $N \otimes_R K \to M \otimes_R K \to L$ is surjective, and therefore so is $M \otimes_R K \to L$.

(ii) : Using the commutative square

$$\begin{array}{ccc} M \otimes_R K & \longrightarrow & V \\ \downarrow & & \downarrow \varphi \\ \varphi(M) \otimes_R K & \longrightarrow & V \end{array}$$

we see that the lower horizontal arrow must be surjective. □

LEMMA 3.1.3. *Let M, N be lattices in V. Then:*

(i) The R-submodule $M \cap N$ is a lattice in V.

(ii) The R-module $M/M \cap N$ has finite length.

PROOF. Let $m_1, \cdots, m_n$ be a set of generators of the R-module M. Since N is a lattice in V, we can find elements $a_1, \cdots, a_n \in R - \{0\}$ such that $a_i m_i \in N$ for all $i = 1, \cdots, n$. Writing $a = a_1 \cdots a_n \in R$, we have $aM \subset M \cap N$. Then aM is a lattice in V by Lemma 3.1.2 (ii), and so is $M \cap N$ by Lemma 3.1.2 (i). This proves (i).

We have $\dim M/aM \leq \dim R/aR \leq 0$, hence the R-module M/aM has finite length (Proposition 1.1.3), and so has its quotient $M/M \cap N$, proving (ii). □

DEFINITION 3.1.4. Let M, N be lattices in V. We define

$$d(M, N) = l_R(M/(M \cap N)) - l_R(N/(M \cap N)) \in \mathbb{Z}.$$

One sees immediately that:

- We have $d(M,N) + d(N,M) = 0$.
- If $N \subset M$, then $d(M,N) = l_R(M/N)$.

LEMMA 3.1.5. *Let M, N, P be lattices in V. Then*

$$d(M,N) + d(N,P) = d(M,P).$$

PROOF. Assume first that $P \subset M \cap N$. Then we have exact sequences of R-modules

$$0 \to (M \cap N)/P \to M/P \to M/(M \cap N) \to 0$$

and

$$0 \to (M \cap N)/P \to N/P \to N/(M \cap N) \to 0,$$

so that, using the additivity of the length,

$$\begin{aligned} d(M,N) &= l_R(M/(M \cap N)) - l_R(N/(M \cap N)) \\ &= l_R(M/P) - l_R(N/P) \\ &= d(M,P) - d(N,P), \end{aligned}$$

and the formula is true in this case.

In general (when $P \not\subset M \cap N$), the R-submodule $Q = P \cap M \cap N$ is a lattice in V, by applying twice Lemma 3.1.3 (i). Using three times the case above, we have

$$\begin{aligned} d(M,N) + d(N,P) &= d(M,Q) + d(Q,N) + d(N,Q) + d(Q,P) \\ &= d(M,Q) + d(Q,P) \\ &= d(M,P). \end{aligned}$$

□

LEMMA 3.1.6. *Let φ be a K-automorphism of V. The integer $d(M, \varphi(M))$ does not depend on the lattice M in V.*

PROOF. Let M, N be two lattices in V. Then, by Lemma 3.1.5,

$$d(M, \varphi(M)) = d(M,N) + d(N, \varphi(N)) + d(\varphi(N), \varphi(M)).$$

Since φ induces isomorphisms

$$M/M \cap N \to \varphi(M)/\varphi(M) \cap \varphi(N) \quad \text{and} \quad N/M \cap N \to \varphi(N)/\varphi(M) \cap \varphi(N),$$

we see that

$$d(\varphi(N), \varphi(M)) = d(N,M) = -d(M,N),$$

and the statement follows. □

PROPOSITION 3.1.7. *Let M be a lattice in V, and φ a K-automorphism of V. Then*

$$d(M, \varphi(M)) = \operatorname{ord}_R(\det \varphi).$$

PROOF. Letting $e_1, \cdots, e_n \in M$ be a K-basis of V, in view of Lemma 3.1.6, we may replace M by the lattice $\bigoplus_i Re_i$, and assume that $e_1, \cdots, e_n$ generate M. If ψ is another K-automorphism of V, we have, using Lemma 3.1.5, Lemma 3.1.6 and Lemma 3.1.3 (ii),

$$d(M, \psi \circ \varphi(M)) = d(M, \varphi(M)) + d(\varphi(M), \psi \circ \varphi(M)) = d(M, \varphi(M)) + d(M, \psi(M)).$$

We also have

$$\operatorname{ord}_R(\det(\psi \circ \varphi)) = \operatorname{ord}_R((\det \psi) \cdot (\det \varphi)) = \operatorname{ord}_R(\det \psi) + \operatorname{ord}_R(\det \varphi).$$

Therefore each of the two functions

$$\varphi \mapsto d(M, \varphi(M)) \text{ and } \varphi \mapsto \operatorname{ord}_R(\det \varphi)$$

defines a group homomorphism

$$\operatorname{Aut}_K(V) \to \mathbb{Z}.$$

Since $\operatorname{Aut}_K(V)$ is generated by automorphisms whose matrices in the basis $e_1, \cdots, e_n$ are elementary, we may assume that the matrix of φ is elementary.

If this matrix is permutation then $\varphi(M) = M$. If for some i, j, we have $\varphi(e_k) = e_k$ for all $k \neq i$, and $\varphi(e_i) = e_i + (a/b)e_j$ for some $a, b \in R$ and $j \neq i$, then replacing e_i by be_i (thus modifying M), we may assume that $b = 1$, and therefore $M = \varphi(M)$. In these two cases $\det \varphi = \pm 1 \in R^\times$, and we conclude that

$$d(M, \varphi(M)) = 0 = \operatorname{ord}_R(\det \varphi).$$

Finally assume that the matrix of φ is diagonal, with entries $(1, \cdots, 1, a)$ with $a \in K^\times$. Since we may restrict to a generating set of the group $\operatorname{Aut}_K(V)$, we may assume that $a \in R - \{0\}$. Then $\varphi(M) \subset M$ and

$$M/\varphi(M) = R^{\oplus n}/(R^{\oplus n-1} \oplus aR) = R/aR,$$

so that $d(M, \varphi(M)) = l(R/aR)$. But $\det \varphi = a$, hence $\operatorname{ord}_R(\det \varphi) = \operatorname{ord}_R(a) = l(R/aR)$, as required. □

2. Proper push-forward of principal divisors

PROPOSITION 3.2.1. *Let $f: Y \to X$ be a proper and surjective morphism. Assume that Y and X are integral, and that $\dim Y = \dim X$. Then for any $\varphi \in k(Y)^\times$, we have*

$$f_* \circ \operatorname{div} \varphi = \operatorname{div}\left(N_{k(Y)/k(X)}(\varphi)\right) \in \mathcal{Z}(X),$$

where $N_{k(Y)/k(X)}: k(Y)^\times \to k(X)^\times$ is the norm of the field extension.

PROOF. — *Case f is finite.* Let $x \in X$ be a point of codimension one. We compare the coefficients at x on the two sides of the equation. Letting $A = \mathcal{O}_{X,x}$. The scheme $f^{-1} \operatorname{Spec} A$ can be written as $\operatorname{Spec} B$, since it is finite over $\operatorname{Spec} A$. We have $\dim A = \dim B = 1$. Writing φ as quotient of elements of B, we may assume that $\varphi \in B$. The points $y \in Y$ such that $f(y) = x$ are in bijective correspondence with the maximal ideals $\mathfrak{q}$ of B. On the left hand side, we have (here $\mathfrak{q}$ runs over the maximal ideals of B)

$$\begin{aligned}\sum_{y \in f^{-1}\{x\}} [k(y) : k(x)] \operatorname{ord}_y(\varphi) &= \sum_{\mathfrak{q}} [B/\mathfrak{q} : A/\mathfrak{m}_A] l(B_\mathfrak{q}/\varphi B_\mathfrak{q}) \\ &= \sum_{\mathfrak{q}} l_A(B/\mathfrak{q}) l(B_\mathfrak{q}/\varphi B_\mathfrak{q}) \\ &= l_A(B/\varphi B),\end{aligned}$$

where we used Lemma 1.5.11 with $M = B/\varphi B$ for the last equality.

On the right hand side, the coefficient at x is

$$\operatorname{ord}_x(\det m_\varphi)$$

where m_φ is the multiplication by φ in the $k(X)$-algebra $k(Y)$. We apply Proposition 3.1.7 and Example 3.1.1 for the ring $R = A$, the lattice B in $V = k(Y)$.

— *Case f is birational and X is normal.* Let $x \in X$ be a point of codimension one. Let $y \in Y$ be such that $f(y) = x$. Then $\mathcal{O}_{X,x} \subset \mathcal{O}_{Y,y}$ is a local morphism and $\mathcal{O}_{X,x}$ is a valuation ring of $k(X)$ (it is a discrete valuation ring, being a local integrally closed domain of dimension one). Thus $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$ as subrings of $k(X)$ and in particular y has codimension one in Y. This proves that the points $y \in Y$ such that $f(y) = x$ are in bijective correspondence with the morphisms $\operatorname{Spec} \mathcal{O}_{X,x} \to Y$ over X, and by the valuative criterion of properness there is exactly one such morphism. Thus $f^{-1}\{x\} = \{y\}$ for some $y \in Y$. From the equality $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$, we deduce that $[k(y) : k(x)] = 1$, and that the component of $\operatorname{div} \varphi \in \mathcal{Z}(Y)$ at y is the same as the component $\operatorname{div} \varphi \in \mathcal{Z}(X)$ at x. Therefore $f_* \circ \operatorname{div} \varphi = \operatorname{div} \varphi$, as required in this case.

— *General case.* Let $Y' \to Y$ be the normalisation of Y (in $k(Y)$), and $X' \to X$ the normalisation of X in $k(Y)$. By the universal property of the normalisation, the dominant morphism f lifts to a dominant morphism $Y' \to X'$. We may view φ as element of $k(Y')^\times = k(Y)^\times$. Since the morphisms $X' \to X$ and $Y' \to Y$ are finite (we are working with varieties, which are of finite type over a field), and $Y' \to X'$ is a birational morphism with normal target, we conclude using the two case considered above. □

COROLLARY 3.2.2. *Let $f \colon Y \to X$ be a proper surjective morphism between integral varieties, and $\varphi \in k(X)^\times$. Then, using Definition 1.2.3,*

$$f_* \circ \operatorname{div}(f^*\varphi) = \deg(Y/X) \cdot \operatorname{div} \varphi \in \mathcal{Z}(X),$$

PROOF. Let $d = \deg(Y/X)$. Assume that $\dim Y = \dim X$. Then the norm of $f^*\varphi \in k(Y)^\times$ is $\varphi^d \in k(X)^\times$, and we have by Proposition 3.2.1,

$$f_* \circ \operatorname{div}(f^*\varphi) = \operatorname{div}(\varphi^d) = d \operatorname{div}(\varphi) = d \cdot \operatorname{div}(\varphi) \in \mathcal{Z}(X).$$

Now assume that $\dim Y > \dim X$, and let W be an integral closed subvariety of codimension one in Y. If $f(W) = X$, then the inclusion $k(X) \to k(Y)$ factors through $\mathcal{O}_{Y,W}$, and in particular $f^*\varphi \in (\mathcal{O}_{Y,W})^\times \subset k(Y)^\times$, so that $\operatorname{ord}_W(f^*\varphi) = 0$. If $f(W) \neq X$, then $\dim W > \dim f(W)$, and $f_*[W] = 0 \in \mathcal{Z}(X)$. Thus

$$f_* \circ \operatorname{div}(f^*\varphi) = \sum_W f_*\big(\operatorname{ord}_W(f^*\varphi) \cdot [W]\big) = 0,$$

where W runs over the integral closed subvarieties of codimension one in Y □

The next statement is nontrivial only in case $\dim X = 1$.

LEMMA 3.2.3. *Let X be a integral variety, proper over* $\operatorname{Spec} k$, *and $\varphi \in k(X)^\times$. Then, using the notation of Example 1.2.5*

$$\deg \circ \operatorname{div} \varphi = 0.$$

PROOF. Sending t to φ gives rise to a k-scheme morphism $\operatorname{Spec} k(X) \to \operatorname{Spec} k[t, t^{-1}] = \mathbb{P}^1 - \{0, \infty\}$, and therefore a morphism $\operatorname{Spec} k(X) \to X \times_k \mathbb{P}^1$. Let Z be its scheme-theoretic image; this is an integral closed subscheme of $X \times_k \mathbb{P}^1$. The image P of Z in $\mathbb{P}^1$

is closed, by the properness of X over k. Thus we have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{h} & \operatorname{Spec} k \end{array}$$

where each morphism if proper and surjective, and f is additionally birational. Since P is not contained in $\{0, \infty\}$, the element t maps to an element $\pi \in k(P)^\times$. By construction $f^*\varphi = p^*\pi \in k(Z)^\times$. Thus

$$\begin{aligned} g_* \circ \operatorname{div} \varphi &= g_* \circ f_* \circ \operatorname{div}(f^*\varphi) && \text{by Corollary 3.2.2} \\ &= g_* \circ f_* \circ \operatorname{div}(p^*\pi) && \\ &= h_* \circ p_* \circ \operatorname{div}(p^*\pi) && \text{by Lemma 1.2.6} \\ &= \deg(Z/P) \cdot h_* \circ \operatorname{div} \pi && \text{by Corollary 3.2.2.} \end{aligned}$$

Now either $\dim P = 0$ or $P = \mathbb{P}^1$. In the first case $\operatorname{div} \pi \in \mathcal{Z}_{-1}(P) = 0$. If $P = \mathbb{P}^1$, then $\pi = t \in k(\mathbb{P}^1)^\times$, and we have in $\mathcal{Z}(\mathbb{P}^1)$

$$h_* \circ \operatorname{div} \pi = h_*([0] - [\infty]) = [k(0) : k] - [k(\infty) : k] = 0. \qquad \square$$

Lemma 3.2.3 says that, when X is a complete variety, the degree map of Example 1.2.5 descends to a group homomorphism

$$\deg\colon \operatorname{CH}(X) \to \mathbb{Z}.$$

Theorem 3.2.4. *Let $f\colon Y \to X$ be a proper morphism. Then $f_*\mathcal{R}(Y) \subset \mathcal{R}(X)$, which gives a group homomorphism*

$$f_*\colon \operatorname{CH}_\bullet(Y) \to \operatorname{CH}_\bullet(X).$$

Proof. As already observed, the statement is certainly true when f is a closed immersion. Thus we may assume that X, Y are integral and f surjective, take $\varphi \in k(Y)^\times$ and prove that $f_* \circ \operatorname{div} \varphi \in \mathcal{R}(X)$. If $\dim Y = \dim X$, the result follows from Proposition 3.2.1. If $\dim Y > \dim X + 1$, then $f_* \circ \operatorname{div} \varphi \in \mathcal{Z}_{\dim Y - 1}(X) = 0$. Thus we may assume that $\dim Y = \dim X + 1$. Then $f_* \circ \operatorname{div} \varphi = d \cdot [X]$, where

$$d = \sum_y [k(y) : k(X)] \operatorname{ord}_y(\varphi),$$

and y runs over the set of points of codimension one in Y such that $f(y)$ is the generic point of X. The generic fiber $F = Y \times_X \operatorname{Spec} k(X)$ is an integral $k(X)$-variety, and letting ψ be the image of φ under the isomorphism $k(Y)^\times \simeq k(F)^\times$, we have $d = \deg \circ \operatorname{div} \psi$. This integer vanishes, by Lemma 3.2.3 applied to the $k(X)$-variety F. $\square$

CHAPTER 4

Divisor classes

1. The divisor attached to a meromorphic section

Let X be a variety. An $\mathcal{O}_X$-module will be called *invertible* if it is locally free of rank one, i.e. if each point of X is contained in an open subscheme U such that $\mathcal{L}$ restricts to a free $\mathcal{O}_U$-module of rank one on U.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. When $i: V \to X$ is a closed or open immersion, we denote by $\mathcal{L}|_V$ the invertible $\mathcal{O}_V$-module $i^*\mathcal{L}$.

Definition 4.1.1. Assume that X is integral, with generic point η. A *regular meromorphic section* of $\mathcal{L}$ is an nonzero element of the generic stalk of $\mathcal{L}$, i.e. an element of $\mathcal{L}_\eta - \{0\}$. The set of regular meromorphic sections of $\mathcal{L}$ is noncanonically in bijection with $k(X)^\times$. When s, t are two regular meromorphic sections $\mathcal{L}$, we write $s/t \in k(X)^\times$ for the unique element such that $(s/t) \cdot t = s$.

Let x be a point of codimension one in X, and $u \in \mathcal{L}_x$ a generator of the free $\mathcal{O}_{X,x}$-module $\mathcal{L}_x$. We may view u as a regular meromorphic section of $\mathcal{L}$, via the injection $\mathcal{L}_x \to \mathcal{L}_\eta$. The integer

$$\operatorname{ord}_{\mathcal{L},x}(s) = \operatorname{ord}_x(s/u).$$

does not depend on the choice of u. Indeed, if $u' \in \mathcal{L}_x$ is another generator, then $u = \lambda \cdot u'$ for some $\lambda \in (\mathcal{O}_{X,x})^\times$. Therefore

$$s = (s/u) \cdot u = \lambda \cdot (s/u) \cdot u'$$

so that $s/u' = \lambda \cdot (s/u)$, and

$$\operatorname{ord}_x(s/u') = \operatorname{ord}_x(\lambda \cdot (s/u)) = \operatorname{ord}_x(\lambda) + \operatorname{ord}_x(s/u) = \operatorname{ord}_x(s/u).$$

When $\mathcal{L} = \mathcal{O}_X$, the regular meromorphic section s corresponds to an element $\varphi \in k(X)^\times$, and we have

$$\operatorname{ord}_{\mathcal{L},x}(s) = \operatorname{ord}_x(\varphi) \in \mathbb{Z}. \tag{4.1.c}$$

Lemma 4.1.2. *Let X be an integral variety, $\alpha: \mathcal{L} \to \mathcal{M}$ an isomorphism of invertible $\mathcal{O}_X$-modules, and s a regular meromorphic section of $\mathcal{L}$. Then for any point x of codimension one in X, we have*

$$\operatorname{ord}_{\mathcal{L},x}(s) = \operatorname{ord}_{\mathcal{M},x}(\alpha(s)).$$

Proof. Let η be the generic point of X, and u a generator of $\mathcal{L}_x$. Then $\alpha(u)$ is a generator of $\mathcal{M}_x$, and

$$\alpha(s) = \alpha((s/u) \cdot u) = (s/u) \cdot \alpha(u) \in \mathcal{M}_\eta,$$

so that $\alpha(s)/\alpha(u) = s/u$, and

$$\operatorname{ord}_{\mathcal{M},x}(\alpha(s)) = \operatorname{ord}_x(\alpha(s)/\alpha(u)) = \operatorname{ord}_x(s/u) = \operatorname{ord}_{\mathcal{L},x}(s).$$

□

LEMMA 4.1.3. *Let X be an integral variety, and s a regular meromorphic section of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Then the set of points x of codimension one in X such that* $\operatorname{ord}_{\mathcal{L},x}(s) \neq 0$ *is finite.*

PROOF. Taking a finite cover of X by affine open subschemes where the restriction of $\mathcal{L}$ is trivial, this follows from Lemma 4.1.2, (4.1.c) and Proposition 2.1.3 □

DEFINITION 4.1.4. Let X be an integral variety, $\mathcal{L}$ an invertible $\mathcal{O}_X$, and s a regular meromorphic section of $\mathcal{L}$. We define

$$\operatorname{div}_{\mathcal{L}}(s) = \sum_V \operatorname{ord}_{\mathcal{L},\eta_V}(s)[V] \in \mathcal{Z}(X),$$

where V runs over the integral closed subvarieties of codimension one in X, and η_V denotes the generic point of V.

LEMMA 4.1.5. *Let X be an integral variety, and $\mathcal{L}, \mathcal{M}$ invertible $\mathcal{O}_X$-modules.*

(i) Let $\alpha\colon \mathcal{L} \to \mathcal{M}$ be an isomorphism, and s a regular meromorphic section of $\mathcal{L}$. Then

$$\operatorname{div}_{\mathcal{L}}(s) = \operatorname{div}_{\mathcal{M}}(\alpha(s)).$$

(ii) Let $\varphi \in k(X)^\times$. Then, viewing φ as a regular meromorphic section of $\mathcal{O}_X$,

$$\operatorname{div}_{\mathcal{O}_X}(\varphi) = \operatorname{div} \varphi.$$

(iii) Let s, resp. t, be a regular meromorphic section of $\mathcal{L}$, resp. $\mathcal{M}$. Then

$$\operatorname{div}_{\mathcal{L}\otimes\mathcal{M}}(s \otimes t) = \operatorname{div}_{\mathcal{L}}(s) + \operatorname{div}_{\mathcal{M}}(t).$$

(iv) Let s, t be two regular meromorphic sections of $\mathcal{L}$. Then

$$\operatorname{div}_{\mathcal{L}}(s) = \operatorname{div}_{\mathcal{L}}(t) + \operatorname{div}(s/t).$$

PROOF. (ii) follows from (4.1.c), and (i) from Lemma 4.1.2.

To prove (iii), let x be a point of codimension one in X, u a generator of $\mathcal{L}_x$, and v a generator of $\mathcal{M}_x$. Then

$$s \otimes t = \big((s/u)\cdot u\big) \otimes \big((t/v)\cdot v\big) = (s/u)\cdot(t/v)\cdot u \otimes v,$$

and therefore

$$\begin{aligned}\operatorname{ord}_{\mathcal{L}\otimes\mathcal{M},x}(s\otimes t) &= \operatorname{ord}_x((s\otimes t)/(u \otimes v))\\ &= \operatorname{ord}_x((s/u)\cdot(t/v))\\ &= \operatorname{ord}_x(s/u) + \operatorname{ord}_x(t/v)\\ &= \operatorname{ord}_{\mathcal{L},x}(s) + \operatorname{ord}_{\mathcal{M},x}(t),\end{aligned}$$

and (iii) follows.

(iv) may be proved simarly, but in fact follows from (i), (ii), (iii). □

Let $f\colon Y \to X$ be a dominant morphism between integral varieties, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Let ξ and η be the respective generic points of Y and X. There is a canonical identification

$$\mathcal{L}_\eta \otimes_{k(X)} k(Y) = (f^*\mathcal{L})_\xi.$$

Let s a regular meromorphic section of $\mathcal{L}$. Then $s \otimes 1$ corresponds to a regular meromorphic section f^*s of $f^*\mathcal{L}$.

When $\mathcal{L} = \mathcal{O}_X$, the regular meromorphic section s corresponds to an element $\varphi \in k(X)^\times$. Then the regular meromorphic section f^*s corresponds to $f^*\varphi \in k(Y)^\times$.

LEMMA 4.1.6. *Let $f\colon Y \to X$ be a proper surjective morphism between integral varieties, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and s a regular meromorphic section of $\mathcal{L}$. Then, using Definition 1.2.3,*

$$f_* \circ \operatorname{div}_{f^*\mathcal{L}}(f^*s) = \deg(Y/X) \cdot \operatorname{div}_{\mathcal{L}}(s) \in \mathcal{Z}(X),$$

PROOF. The question is local on X, and we may assume given an isomorphism $\mathcal{L} \to \mathcal{O}_X$. Then the result follows Lemma 4.1.5, (i), (ii) and Corollary 3.2.2. □

LEMMA 4.1.7. *Let $f\colon Y \to X$ be a flat morphism having a relative dimension, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Assume that X is integral, and let s be a regular meromorphic section of $\mathcal{L}$. Then*

$$f^* \circ \operatorname{div}_{\mathcal{L}}(s) = \sum_i m_i \operatorname{div}_{f_i^*\mathcal{L}}(f_i^*s) \in \mathcal{Z}(Y),$$

where $m_i = l(\mathcal{O}_{Y,Y_i})$ are the multiplicities of the irreducible components of Y_i of Y, and $f_i\colon Y_i \to X$ the restrictions of f.

PROOF. The question is local on X, and we may assume given an isomorphism $\mathcal{L} \to \mathcal{O}_X$. Then the result follows Lemma 4.1.5 (i) (ii) and Lemma 2.2.1. □

2. The first Chern class

Let now X be a (possibly nonintegral) variety, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that V is an integral closed subscheme of X, and choose a regular meromorphic section s of $\mathcal{L}|_V$. The class of $\operatorname{div}_{\mathcal{L}|_V}(s) \in \mathrm{CH}(X)$ does not depend on the choice of s by Lemma 4.1.5 (iv). We obtain a group homomorphism

$$c_1(\mathcal{L})\colon\ \mathcal{Z}_\bullet(X) \to \mathrm{CH}_{\bullet-1}(X).$$

PROPOSITION 4.2.1. *Let $\mathcal{L}, \mathcal{M}$ be invertible $\mathcal{O}_X$-modules. Then*

(i) If $\mathcal{L} \simeq \mathcal{M}$, then $c_1(\mathcal{L}) = c_1(\mathcal{M})$.
(ii) We have $c_1(\mathcal{L} \otimes \mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M})$.
(iii) We have $c_1(\mathcal{O}_X) = 0$.

PROOF. This follows from Lemma 4.1.5. □

PROPOSITION 4.2.2. *Let $f\colon Y \to X$ be a proper morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then*

$$f_* \circ c_1(f^*\mathcal{L}) = c_1(\mathcal{L}) \circ f_*\colon\ \mathcal{Z}(Y) \to \mathrm{CH}(X).$$

PROOF. The statement is true when f is closed embedding by construction of $c_1(\mathcal{L})$. Thus it will suffice to prove that

$$f_* \circ c_1(f^*\mathcal{L})[Y] = c_1(\mathcal{L}) \circ f_*[Y]$$

when f is surjective, and Y and X are integral. Since $f_*[Y] = \deg(Y/X) \cdot [X]$, the statement follows by choosing a regular meromorphic section s of $\mathcal{L}$, and applying Lemma 4.1.6. □

PROPOSITION 4.2.3. *Let $f\colon Y \to X$ be a flat morphism having a relative dimension, and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then*

$$f^* \circ c_1(\mathcal{L}) = c_1(f^*\mathcal{L}) \circ f^*\colon\ \mathcal{Z}(X) \to \mathrm{CH}(Y).$$

PROOF. By Proposition 1.5.9, it will suffice to prove that

$$f^* \circ c_1(\mathcal{L})[X] = c_1(f^*\mathcal{L})[Y] \in \mathrm{CH}(Y)$$

under the additional assumption that X is integral. After choosing a regular meromorphic section s of $\mathcal{L}$, this follows from Lemma 4.1.7. □

3. Effective Cartier divisors II

Let X be a variety and $D \to X$ an effective Cartier divisor. We denote by $\mathcal{O}(D)$ the invertible $\mathcal{O}_X$-module $(\mathcal{I}_D)^\vee$, defined as the dual of the ideal defining D in X. The natural morphism $\mathcal{I}_D \to \mathcal{O}_X$ is then a global section 1_D of the $\mathcal{O}_X$-module $\mathcal{O}(D)$. If X is integral, the section 1_D is nonzero at the generic point of X, and we may view 1_D as a regular meromorphic section of $\mathcal{O}(D)$.

If $f\colon Y \to X$ is a morphism such that $f^{-1}D \to Y$ is an effective Cartier divisor, then $f^*\mathcal{O}(D) = \mathcal{O}(f^{-1}D)$. To see this, note that the image $\mathcal{I}_{f^{-1}D}$ of the morphism $f^*\mathcal{I}_D \to \mathcal{O}_Y$ is an invertible $\mathcal{O}_Y$-module, and so is its source. This morphism is injective, since a surjection between locally free modules of the same rank is necessarily an isomorphism.

This is so in particular when $f\colon Y \to X$ is a dominant morphism between integral varieties. In this case, we have defined the pull-back f^*1_D, and we have $1_{f^{-1}D} = f^*(1_D)$ as regular meromorphic sections of $\mathcal{O}(f^{-1}D) = f^*\mathcal{O}(D)$.

LEMMA 4.3.1. *Let X be an integral variety and $D \to X$ an effective Cartier divisor. Then*

$$\mathrm{div}_{\mathcal{O}(D)}(1_D) = [D] \in \mathcal{Z}(X),$$

PROOF. Let x be a point of codimension one in X, and a a generator of the $\mathcal{O}_{X,x}$-module $\mathcal{I}_{D,x}$. The effective Cartier divisor D is defined at the point x by the image $b = 1_D(a)$ of a under the morphism $1_D\colon \mathcal{I}_D \to \mathcal{O}_X$. The coefficient of $[D] \in \mathcal{Z}(X)$ at x is

$$l(\mathcal{O}_{X,x}/b\mathcal{O}_{X,x}) = \mathrm{ord}_x(b).$$

On the other hand, the element $b \in \mathcal{O}_{X,x}$ is also the image of $1_D \otimes a$ under the isomorphism $\mathcal{O}(D) \otimes \mathcal{I}_D \to \mathcal{O}_X$. Thus, using Lemma 4.1.5 (i) (iii), we have in $\mathcal{Z}(X)$

$$\mathrm{ord}_x(b) = \mathrm{ord}_{\mathcal{O}(D)\otimes\mathcal{I}_D,x}(1_D \otimes a) = \mathrm{ord}_{\mathcal{O}(D),x}(1_D) + \mathrm{ord}_{\mathcal{I}_D,x}(a).$$

Since $\mathrm{ord}_{\mathcal{I}_D,x}(a) = \mathrm{ord}_x(a/a) = 0$, the statement is proved. □

PROPOSITION 4.3.2. *Let $f\colon Y \to X$ be a proper surjective morphism between integral varieties, and $D \to X$ an effective Cartier divisor. Then*

$$f_*[f^{-1}D] = \deg(Y/X) \cdot [D] \in \mathcal{Z}(D).$$

PROOF. It suffices to prove the equality in $\mathcal{Z}(X)$. We have

$$\begin{aligned}
f_*[f^{-1}D] &= f_* \circ \mathrm{div}_{\mathcal{O}(f^{-1}D)}(1_{f^{-1}D}) && \text{by Lemma 4.3.1} \\
&= f_* \circ \mathrm{div}_{f^*\mathcal{O}(D)}(f^*1_D) && \\
&= \deg(Y/X) \cdot \mathrm{div}_{\mathcal{O}(D)}(1_D) && \text{by Lemma 4.1.6} \\
&= \deg(Y/X) \cdot [D] && \text{by Lemma 4.3.1 .}
\end{aligned}$$

□

4. Intersecting with effective Cartier divisors

The *support* $|\alpha|$ of a cycle $\alpha \in \mathcal{Z}(X)$ is the union of the integral closed subschemes V of X such that the coefficient of α at V is non-zero. This is a closed subset of X, since there are only finitely many such V's.

Definition 4.4.1. Let $D \to X$ be an effective Cartier divisor. Let V an integral closed subscheme of X of dimension n. If $V \not\subset D$, the closed embedding $D \cap V \to V$ is an effective Cartier divisor, hence $D \cap V$ has pure dimension $n-1$, and we let

$$D \cdot [V] = [D \cap V] \in \mathrm{CH}_{n-1}(D \cap V).$$

If $V \subset D$, then we let

$$D \cdot [V] = c_1(\mathcal{O}(D)|_V)[V] \in \mathrm{CH}_{n-1}(V) = \mathrm{CH}_{n-1}(D \cap V).$$

Now for an arbitrary cycle

$$\alpha = \sum_V m_V [V] \in \mathcal{Z}_n(X)$$

where V runs over integral closed subschemes of X of dimension n, and $m_V \in \mathbb{Z}$ (nonzero for only finitely many V's), we define

$$D \cdot \alpha = \sum_V m_V D \cdot [V] \in \mathrm{CH}_{n-1}(D \cap |\alpha|).$$

In order to improve readability, we will often omit to mention the push-forwards along closed embeddings.

Lemma 4.4.2. *Let $D \to X$ be an effective Cartier divisor. If X is equidimensional, then*

$$D \cdot [X] = [D] \in \mathrm{CH}(D).$$

Proof. This is a reformulation of Proposition 1.3.5. □

Lemma 4.4.3. *Let $D \to X$ be an effective Cartier divisor, and $\alpha \in \mathcal{Z}(X)$. Then*

$$c_1(\mathcal{O}(D)|_{|\alpha|})(\alpha) = D \cdot \alpha \in \mathrm{CH}(|\alpha|).$$

Proof. We may assume that $\alpha = [V]$ for an integral closed subscheme V of X. If $V \subset D$, then the statement is true by Definition 4.4.1. If $V \not\subset D$, then $V \cap D \to V$ is an effective Cartier divisor, and the statement follows from Lemma 4.3.1 and the definition of the first Chern class of a line bundle. □

Proposition 4.4.4. *Let $f : Y \to X$ be a proper morphism and $D \to X$ an effective Cartier divisor. Let $\alpha \in \mathcal{Z}(Y)$. Denote by $h : (f^{-1}D) \cap |\alpha| \to D \cap f(|\alpha|)$ the induced morphism. If $f^{-1}D \to Y$ is an effective Cartier divisor, then*

$$h_*((f^{-1}D) \cdot \alpha) = D \cdot f_*\alpha \in \mathrm{CH}(D \cap f(|\alpha|)).$$

Proof. It will suffice to consider the case when $\alpha = [W]$, for W an integral closed subscheme of Y. Let $V = f(W)$. If $W \subset f^{-1}D$, then by Proposition 4.2.2, we have in

$\mathrm{CH}(V) = \mathrm{CH}(D \cap V)$,

$$\begin{aligned} h_*((f^{-1}D) \cdot [W]) &= h_* \circ c_1(\mathcal{O}(f^{-1}D)|_W)[W] \\ &= h_* \circ c_1(h^*(\mathcal{O}(D)|_V))[W] \\ &= c_1(\mathcal{O}(D)|_V) \circ h_*[W] && \text{by Proposition 4.2.2} \\ &= D \cdot h_*[W] && \text{by Lemma 4.4.3.} \end{aligned}$$

If $W \not\subset f^{-1}D$, then $V \not\subset D$, and we have in $\mathrm{CH}(D \cap V)$

$$\begin{aligned} h_*((f^{-1}D) \cdot [W]) &= h_*[(f^{-1}D) \cap W] \\ &= h_*[f^{-1}(D \cap V)] \\ &= \deg(W/V)[D \cap V] && \text{by Proposition 4.3.2} \\ &= D \cdot (\deg(W/V)[V]) \\ &= D \cdot h_*[W]. \end{aligned}$$

□

PROPOSITION 4.4.5. *Let $f\colon Y \to X$ be a flat morphism having a relative dimension. Let $D \to X$ be an effective Cartier divisor, and $\alpha \in \mathcal{Z}(X)$. Denote by $h\colon f^{-1}(D \cap |\alpha|) \to D \cap |\alpha|$ the induced morphism. Then*

$$h^*(D \cdot \alpha) = (f^{-1}D) \cdot f^*\alpha \in \mathrm{CH}(f^{-1}(D \cap |\alpha|)).$$

PROOF. It will suffice to consider the case when $\alpha = [V]$, for V an integral closed subscheme of X. Let $W = f^{-1}V$. If $V \subset D$, then by Proposition 4.2.3, we have in $\mathrm{CH}(W) = \mathrm{CH}(f^{-1}(D \cap V))$,

$$\begin{aligned} h^*(D \cdot [V]) &= h^* \circ c_1(\mathcal{O}(D))[V] \\ &= c_1(h^*\mathcal{O}(D)) \circ h^*[V] \\ &= c_1(h^*\mathcal{O}(D))[W] \\ &= c_1(\mathcal{O}(f^{-1}D))[W] \\ &= (f^{-1}D) \cdot [W]. \end{aligned}$$

the last equality holding because $W \subset f^{-1}D$.

If $V \not\subset D$, then $D \cap V \to V$ is an effective Cartier divisor. Since f is flat $(f^{-1}D) \cap W \to W$ is again an effective Cartier divisor (Proposition 1.3.2). We have in $\mathrm{CH}(f^{-1}D \cap W) = \mathrm{CH}(f^{-1}(D \cap V)$

$$\begin{aligned} h^*(D \cdot [V]) &= h^*[D \cap V] && \text{since } V \not\subset D \\ &= [h^{-1}(D \cap V)] && \text{by Lemma 1.5.6} \\ &= [(f^{-1}D) \cap W] \\ &= (f^{-1}D) \cdot [W] && \text{by Lemma 4.4.2} \\ &= (f^{-1}D) \cdot f^*[V]. \end{aligned}$$

□

CHAPTER 5

Commutativity of divisor classes

1. Herbrand quotients II

Definition 5.1.1. Let A be a (commutative noetherian) ring, M a finitely generated A-module, and $a, b \in A$. Assume that $abM = 0$. If the A-modules $M\{a\}/bM$ and $M\{b\}/aM$ have finite length (recall that $M\{a\}$ denotes the a-torsion submodule of M), we define the integer

$$e_A(M, a, b) = l_A(M\{a\}/bM) - l_A(M\{b\}/aM).$$

Otherwise, we set $e_A(M, a, b) = \infty$.

Observe that if $e_A(M, a, b) < \infty$,

- $e_A(M, a, b) = -e_A(M, b, a)$.
- If $a = 0$, then $e_A(M, a, b) = e_A(M, b)$ (see Definition 1.4.2).

Lemma 5.1.2. *If the A-module M has finite length, then $e_A(M, a, b) = 0$.*

Proof. We have an exact sequence of A-modules of finite length

$$0 \to M\{a\}/bM \to M/bM \xrightarrow{a} M\{b\} \to M\{b\}/aM \to 0,$$

hence $e_A(M, a, b) = e_A(M, b)$, which vanishes by Lemma 1.4.3. □

Lemma 5.1.3. *Consider an exact sequence of finitely generated A-modules*

$$0 \to M' \to M \to M'' \to 0.$$

such that $abM = 0$. If two if the three $e_A(M, a, b), e_A(M', a, b), e_A(M'', a, b)$ are finite, then so is the third, and

$$e_A(M, a, b) = e_A(M', a, b) + e_A(M'', a, b).$$

Proof. If N is an A-module such that $abN = 0$, then multiplication with b induces an exact sequence of A-modules

$$0 \to N\{b\}/aN \to N/aN \to N\{a\} \to N\{a\}/bN \to 0.$$

By the snake lemma, we obtain an exact sequence of A-modules

$$M'\{a\}/bM' \to M\{a\}/bM \xrightarrow{v} M''\{a\}/bM'' \to M'\{b\}/aM' \xrightarrow{u} M\{b\}/aM \to M''\{b\}/aM''$$

and in particular $\ker u \simeq \operatorname{coker} v$. Exchanging the roles of a and b, we obtain an exact sequence of A-modules

$$M'\{b\}/aM' \xrightarrow{u} M\{b\}/aM \to M''\{b\}/aM'' \to M'\{a\}/bM' \to M\{a\}/bM \xrightarrow{v} M''\{a\}/bM''.$$

The statements follow. □

LEMMA 5.1.4. *Let $M \to N$ be a morphism of finitely generated A-modules whose kernel and cokernel have finite length. If $abM = 0$ and $abN = 0$, then*

$$e_A(M, a, b) = e_A(N, a, b).$$

PROOF. Letting I be the image of $M \to N$, we have exact sequences

$$0 \to K \to M \to I \to 0$$

$$0 \to I \to N \to C \to 0$$

where K and C have finite length. Thus the statement follows from Lemma 5.1.3 and Lemma 5.1.2. □

LEMMA 5.1.5. *Let M be a finitely generated A-module and $a, b \in A$ such that $abM = 0$. Let $c \in A$ be such that M/cM has finite length. Then*

$$e_A(M, ca, b) = e_A(M, a, b) - e_A(aM, c).$$

PROOF. Let $N \subset M$ be the submodule consisting of those m such that $c^i m = 0$ for some $i \in \mathbb{N}$. Then $(M/N)\{c\} = 0$, so that the module N/cN is a submodule of M/cM, hence has finite length. Its quotient $c^i N/c^{i+1}N$ thus has finite length. Since N is finitely generated, there is j such that $c^j N = 0$. Using the exact sequences for $i = 0, \cdots, j$

$$0 \to c^i N \to c^{i+1}N \to c^i N/c^{i+1}N \to 0$$

we conclude that N has finite length, hence $e_A(N, a, b) = 0$ by Lemma 5.1.2. Thus by Lemma 5.1.3, we have $e_A(M, a, b) = e_A(M/N, a, b)$ and $e_A(M, ca, b) = e_A(M/N, ca, b)$. The kernel of the surjective morphism $M \to a(M/N)$ induced by multiplication with a is a submodule of N, hence has finite length as A-module. Using Lemma 1.4.3 and Lemma 1.4.4, we deduce that $e_A(aM, c) = e_A(a(M/N), c)$. Thus we may replace M with M/N, and therefore assume that $M\{c\} = 0$. We have an exact sequence of A-modules

$$0 \to aM/acM \to M\{b\}/acM \to M\{b\}/aM \to 0.$$

Now since $M\{c\} = 0$, we have $M\{ac\}/bM = M\{a\}/bM$, and using the above exact sequence it follows that $e_A(M, ca, b) < \infty$ if and only if $e_A(M, a, b) < \infty$. In this case,

$$\begin{aligned} e_A(M, ac, b) &= l_A(M\{ac\}/bM) - l_A(M\{b\}/acM) \\ &= l_A(M\{a\}/bM) - l_A(M\{b\}/acM) && \text{since } M\{c\} = 0 \\ &= l_A(M\{a\}/bM) - l_A(M\{b\}/aM) - l_A(aM/acM) \\ &= e_A(M, a, b) - e_A(aM, c), \end{aligned}$$

since $(aM)\{c\} \subset M\{c\} = 0$. □

LEMMA 5.1.6. *Let $x \in A$ and M a finitely generated A-module such that $x^n M = 0$. Then for any $i = 0, \cdots, n$, we have*

$$e_A(M, x^i, x^{n-i}) \in \{0, \infty\}.$$

PROOF. We prove the statement for all modules M by induction on n. If $n = 0$, then $M = 0$, and the statement is true. Assume that $n > 0$. By antisymmetry, we may assume that $2i \leq n$. The statement is clear if $i = 0$ or if $2i = n$. Thus we assume that $e_A(M, x^i, x^{n-i}) \neq \infty$ with $0 < i < n/2$, and prove that $e_A(M, x^i, x^{n-i}) = 0$. For

$j = 0, \cdots, n$, let $M_j = M\{x^j\}$. Observing that $M_{n-2i} \cap x^i M = x^i M_{n-i}$ and $x^{n-i}M = x^{n-2i}(x^i M) \subset x^{n-2i} M_{n-i}$ yields an exact sequence of A-modules

$$0 \to M_{n-2i}/x^i M_{n-i} \to M_{n-i}/x^i M \xrightarrow{x^{n-2i}} M_i/x^{n-i}M \to M_i/x^{n-2i}M_{n-i} \to 0.$$

Since $M_i = M_{n-i}\{x^i\}$ and $M_{n-2i} = M_{n-i}\{x^{n-2i}\}$, additivity of the length function yields

$$e_A(M, x^i, x^{n-i}) = e_A(M_{n-i}, x^i, x^{2n-i}) \in \mathbb{Z}.$$

Appying the induction hypothesis to the module M_{n-i} which satisfies $x^{n-i}M_{n-i} = 0$, we see that this integer vanishes. □

2. The tame symbol

Let A be a discrete valuation ring, with quotient field K and residue field κ. For any $a, b \in K^\times$, the element

$$(-1)^{\operatorname{ord}_A(a) \cdot \operatorname{ord}_A(b)} \cdot a^{\operatorname{ord}_A(b)} \cdot b^{-\operatorname{ord}_A(a)} \in K^\times$$

belongs to $A^\times$ (its valuation is zero). We define an element of $\kappa^\times$ as

$$\partial_A(a, b) = (-1)^{\operatorname{ord}_A(a) \cdot \operatorname{ord}_A(b)} \cdot a^{\operatorname{ord}_A(b)} \cdot b^{-\operatorname{ord}_A(a)} \mod \mathfrak{m}_A.$$

Observe that:

- The map $\partial_A \colon K^\times \times K^\times \to \kappa^\times$ is bilinear and antisymmetric.
- If $a \in A^\times$, then $\partial_A(a, b) = a^{\operatorname{ord}_A(b)}$.
- If $a, b \in A^\times$, then $\partial_A(a, b) = 1$.

THEOREM 5.2.1. *Let A be an integrally closed local domain of dimension two, with quotient field K. Let $a, b \in K^\times$. Then*

$$\sum_{\mathfrak{p}} \operatorname{ord}_{A/\mathfrak{p}} \circ \partial_{A_\mathfrak{p}}(a, b) = 0$$

where $\mathfrak{p}$ runs over the height one primes of A.

This theorem will be proved after a series of lemmas. By bilinearity of $\partial_{A_\mathfrak{p}}$ and linearity of $\operatorname{ord}_{A/\mathfrak{p}}$, it will suffice to prove the theorem under the assumption that $a, b \in A - \{0\}$. Let $B = A/abA$. When $\mathfrak{p}$ is a prime of height one in A, we consider the A-module

$$B(\mathfrak{p}) = \operatorname{im}(B \to B_\mathfrak{p}).$$

LEMMA 5.2.2. *Let $\mathfrak{p}, \mathfrak{q}$ be primes of height one in A. Then*

$$B(\mathfrak{p})_\mathfrak{q} = \begin{cases} 0 & \text{if } \mathfrak{q} \neq \mathfrak{p} \\ B_\mathfrak{p} & \text{if } \mathfrak{q} = \mathfrak{p}. \end{cases}$$

PROOF. By exactness of the localisation at $\mathfrak{q}$, the $A_\mathfrak{q}$-module $B(\mathfrak{p})_\mathfrak{q}$ is the image of the natural morphism $B_\mathfrak{p} \to (B_\mathfrak{p})_\mathfrak{q}$. This morphism is an isomorphism when $\mathfrak{p} = \mathfrak{q}$, and zero when $\mathfrak{p} \not\subset \mathfrak{q}$. □

LEMMA 5.2.3. *Let $\mathfrak{p}$ be a prime of height one in A and $c \in A - \mathfrak{p}$. Then the A-module $B(\mathfrak{p})/cB(\mathfrak{p})$ has finite length.*

PROOF. For any prime $\mathfrak{q}$ of height one in A, we have, in view of Lemma 5.2.2

$$(B(\mathfrak{p})/cB(\mathfrak{p}))_{\mathfrak{q}} = B(\mathfrak{p})_{\mathfrak{q}}/cB(\mathfrak{p})_{\mathfrak{q}} = \begin{cases} 0 & \text{if } \mathfrak{q} \neq \mathfrak{p} \\ B_{\mathfrak{p}}/cB_{\mathfrak{p}} & \text{if } \mathfrak{q} = \mathfrak{p}. \end{cases}$$

Multiplication with $c \in A - \mathfrak{p}$ is an isomorphism on the $A_{\mathfrak{p}}$-module $B_{\mathfrak{p}}$, hence $B_{\mathfrak{p}}/cB_{\mathfrak{p}} = 0$. This proves that the A-module M/cM has support contained in $\{\mathfrak{m}_A\}$, hence finite length (being finitely generated). □

LEMMA 5.2.4. *There are only finitely many primes $\mathfrak{p}$ of height one in A such that $B(\mathfrak{p}) \neq 0$.*

PROOF. There are only finitely many primes $\mathfrak{p}$ of height one in A such that $B_{\mathfrak{p}} \neq 0$: they correspond to the irreducible components of the effective Cartier divisor defined by the ideal abA in $\operatorname{Spec} A$ (or equivalently to those points x of codimension one in $\operatorname{Spec} A$ such that $\operatorname{ord}_x(ab) \neq 0$). Thus the statement follows from Lemma 5.2.2. □

LEMMA 5.2.5. *The kernel and cokernel of the morphism of A-modules*

$$B \to \bigoplus_{\mathfrak{p}} B(\mathfrak{p})$$

have finite length, where $\mathfrak{p}$ runs over the height one primes of A.

PROOF. The localisation of this morphism at every prime of height one in A is an isomorphism by Lemma 5.2.2. Thus the support of its kernel, resp. cokernel, contains no such prime, which means that it is contained in $\{\mathfrak{m}_A\}$. It is also finitely generated by Lemma 5.2.4, hence has finite length. □

LEMMA 5.2.6. *Let $\mathfrak{p}$ be a prime of height one in A, and $c \in A - \mathfrak{p}$. Then the A-module $aB(\mathfrak{p})/caB(\mathfrak{p})$ has finite length, and*

$$e_A(aB(\mathfrak{p}), c) = \operatorname{ord}_{A_{\mathfrak{p}}}(b) \operatorname{ord}_{A/\mathfrak{p}}(c).$$

PROOF. The A-module $B(\mathfrak{p})/cB(\mathfrak{p})$ has finite length by Lemma 5.2.3, hence the same is true for its quotient $aB(\mathfrak{p})/caB(\mathfrak{p})$. We have

$$\begin{aligned} e_A(aB(\mathfrak{p}), c) &= \sum_{\text{height } \mathfrak{q}=1} l_{A_{\mathfrak{q}}}(aB(\mathfrak{p})_{\mathfrak{q}}) \cdot l_A(A/(\mathfrak{q} + cA)) && \text{by Proposition 1.4.5} \\ &= l_{A_{\mathfrak{p}}}(aB_{\mathfrak{p}}) \cdot l_A(A/(\mathfrak{p} + cA)) && \text{by Lemma 5.2.2} \\ &= l_{A_{\mathfrak{p}}}(aB_{\mathfrak{p}}) \cdot \operatorname{ord}_{A/\mathfrak{p}}(c) \end{aligned}$$

Since a is a nonzero element of the domain $A_{\mathfrak{p}}$, we have isomorphisms of $A_{\mathfrak{p}}$-modules

$$A_{\mathfrak{p}}/bA_{\mathfrak{p}} \simeq aA_{\mathfrak{p}}/abA_{\mathfrak{p}} \simeq aB_{\mathfrak{p}},$$

hence $l_{A_{\mathfrak{p}}}(aB_{\mathfrak{p}}) = l(A_{\mathfrak{p}}/bA_{\mathfrak{p}}) = \operatorname{ord}_{A_{\mathfrak{p}}}(b)$. □

PROPOSITION 5.2.7. *Let $\mathfrak{p}$ be a prime of height one in A. We have*

$$-\operatorname{ord}_{A/\mathfrak{p}} \circ \partial_{A_{\mathfrak{p}}}(a, b) = e_A(B(\mathfrak{p}), a, b).$$

PROOF. We first claim that $e_A(B(\mathfrak{p}), a, b) < \infty$. Indeed for any prime $\mathfrak{q}$ of height one in A, we have by Lemma 5.2.2

$$(B(\mathfrak{p})\{a\}/bB(\mathfrak{p}))_{\mathfrak{q}} = \begin{cases} 0 & \text{if } \mathfrak{q} \neq \mathfrak{p} \\ B_{\mathfrak{p}}\{a\}/bB_{\mathfrak{p}} & \text{if } \mathfrak{q} = \mathfrak{p} \end{cases}$$

But $B_{\mathfrak{p}} = A_{\mathfrak{p}}/abA_{\mathfrak{p}}$ and a is a nonzero element of the domain $A_{\mathfrak{p}}$. Thus an element $x \in A_{\mathfrak{p}}$ satisfies $ax \in abA_{\mathfrak{p}}$ if and only if $x \in bA_{\mathfrak{p}}$. This proves that $B_{\mathfrak{p}}\{a\}/bB_{\mathfrak{p}} = 0$, so that the A-module $B(\mathfrak{p})\{a\}/bB(\mathfrak{p})$ has finite length. Of course, the same is true for $B(\mathfrak{p})\{b\}/aB(\mathfrak{p})$, which proves our claim.

Let $e = \operatorname{ord}_{A_{\mathfrak{p}}}(a)$ and $f = \operatorname{ord}_{A_{\mathfrak{p}}}(b)$. Let $c \in A - \mathfrak{p}$ and $a' = ca$, $B' = A/a'bA$ and $B'(\mathfrak{p}) = \operatorname{im}(B' \to B'_{\mathfrak{p}})$. Then using the elementary properties of the tame symbol ∂

$$\begin{aligned}-\operatorname{ord}_{A/\mathfrak{p}} \circ \partial_{A_{\mathfrak{p}}}(a', b) &= -\operatorname{ord}_{A/\mathfrak{p}}\left(\partial_{A_{\mathfrak{p}}}(a,b)\partial_{A_{\mathfrak{p}}}(c,b)\right)\\ &= -\operatorname{ord}_{A/\mathfrak{p}}\left(\partial_{A_{\mathfrak{p}}}(a,b)c^f\right)\\ &= -\operatorname{ord}_{A/\mathfrak{p}} \circ \partial_{A_{\mathfrak{p}}}(a,b) - f\operatorname{ord}_{A/\mathfrak{p}}(c).\end{aligned}$$

Let I be the kernel of the natural surjective morphism $B' \to B$. Then $cI = 0$. Since $c \in A - \mathfrak{p}$, this implies that $(B')_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism, hence so is $B'(\mathfrak{p}) \to B(\mathfrak{p})$. Therefore

$$\begin{aligned}e_A(B'(\mathfrak{p}), a', b) &= e_A(B(\mathfrak{p}), a', b)\\ &= e_A(B(\mathfrak{p}), a, b) - e_A(aB(\mathfrak{p}), c) \quad \text{by Lemma 5.2.3 and Lemma 5.1.5}\\ &= e_A(B(\mathfrak{p}), a, b) - f\operatorname{ord}_{A/\mathfrak{p}}(c) \quad \text{by Lemma 5.2.6.}\end{aligned}$$

Thus while proving the lemma, we may multiply a with an element of $A - \mathfrak{p}$. By antisymmetry we may also multiply b by such an element. Choose a uniformiser $\pi \in A_{\mathfrak{p}}$. Upon multiplying and dividing by elements of $A - \mathfrak{p}$, we may assume that $\pi \in A$, and that $a = \pi^e, b = \pi^f$. Now we compute using Lemma 5.1.6

$$e_A(B(\mathfrak{p}), a, b) = e_A(B(\mathfrak{p}), \pi^e, \pi^f) = 0.$$

On the other hand, using the definition of the tame symbol,

$$-\operatorname{ord}_{A/\mathfrak{p}} \circ \partial_A(a,b) = -\operatorname{ord}_{A/\mathfrak{p}} \circ \partial_A(\pi^e, \pi^f) = \operatorname{ord}_{A/\mathfrak{p}}((-1)^{ef}) = 0.$$

This concludes the proof of the proposition. □

PROOF OF THEOREM 5.2.1. We now can combine these lemmas:

$$\begin{aligned}e_A(B,a,b) &= e_A\Big(\bigoplus_{\text{height } \mathfrak{p}=1} B(\mathfrak{p}), a, b\Big) && \text{by 5.2.5 and 5.1.4}\\ &= \sum_{\text{height } \mathfrak{p}=1} e_A(B(\mathfrak{p}), a, b) && \text{by 5.1.3}\\ &= -\sum_{\text{height } \mathfrak{p}=1} \operatorname{ord}_{A/\mathfrak{p}} \circ \partial_{A_{\mathfrak{p}}}(a,b) && \text{by 5.2.7 .}\end{aligned}$$

To conclude the proof observe that $e_A(B,a,b) = 0$. Indeed, since a, b are nonzero elements of the domain A, it follows that

$$B\{a\} = bB \quad \text{and} \quad B\{b\} = aB.$$ □

3. Commutativity

THEOREM 5.3.1. *Let X be an integral variety of dimension n.*

(i) Let $\mathcal{L}, \mathcal{M}$ be invertible $\mathcal{O}_X$-modules and s, resp. t, a regular meromorphic section of $\mathcal{L}$, resp. $\mathcal{M}$. Then

$$c_1(\mathcal{L}) \circ \operatorname{div}_{\mathcal{M}}(t) = c_1(\mathcal{M}) \circ \operatorname{div}_{\mathcal{L}}(s) \in \mathrm{CH}_{n-2}(X).$$

(ii) Let $\mathcal{M}$ be an invertible $\mathcal{O}_X$-module and t a regular meromorphic section of $\mathcal{M}$. Let $D \to X$ be an effective Cartier divisor. Then

$$D \cdot \mathrm{div}_{\mathcal{M}}(t) = c_1(\mathcal{M}|_D)[D] \in \mathrm{CH}_{n-2}(D).$$

(iii) Let $D \to X$ and $E \to X$ be effective Cartier divisors. Then

$$D \cdot [E] = E \cdot [D] \in \mathrm{CH}_{n-2}(D \cap E).$$

PROOF. Let us first prove (i). The normalisation $\pi \colon X' \to X$ is a finite birational morphism. By Proposition 4.2.2 and Lemma 4.1.6, we have

$$\pi_* \circ c_1(\pi^*\mathcal{L}) \circ \mathrm{div}_{\mathcal{M}}(\pi^* t) = c_1(\mathcal{L}) \circ \pi_* \circ \mathrm{div}_{\mathcal{M}}(\pi^* t) = c_1(\mathcal{L}) \circ \mathrm{div}_{\mathcal{M}}(t),$$
$$\pi_* \circ c_1(\pi^*\mathcal{M}) \circ \mathrm{div}_{\mathcal{L}}(\pi^* s) = c_1(\mathcal{M}) \circ \pi_* \circ \mathrm{div}_{\mathcal{L}}(\pi^* s) = c_1(\mathcal{M}) \circ \mathrm{div}_{\mathcal{L}}(s).$$

Thus we may replace X with X', and assume that X is normal.

Let $x_1, \cdots, x_p$ be the points of codimension one in X such that $\mathrm{ord}_{\mathcal{L},x_i}(s) \neq 0$ or $\mathrm{ord}_{\mathcal{M},x_i}(t) \neq 0$. For each $i = 1, \cdots, p$, let V_i be the closure of x_i, $A_i = \mathcal{O}_{X,x_i}$, and let s_i, resp. t_i, be a generator of the A_i-module $\mathcal{L}_{x_i}$, resp. $\mathcal{M}_{x_i}$. Then, in $\mathrm{CH}(V_i)$

$$c_1(\mathcal{L})[V_i] = \mathrm{div}_{\mathcal{L}|_{V_i}}(s_i) \quad \text{and} \quad c_1(\mathcal{M})[V_i] = \mathrm{div}_{\mathcal{M}|_{V_i}}(t_i).$$

Write $f_i = s/s_i$ and $g_i = t/t_i$ in $k(X)^\times$ so that

$$\mathrm{ord}_{\mathcal{L},x_i}(s) = \mathrm{ord}_{x_i}(f_i) \quad \text{and} \quad \mathrm{ord}_{\mathcal{M},x_i}(t) = \mathrm{ord}_{x_i}(g_i).$$

We now prove that, in $\mathcal{Z}_{n-2}(X)$

$$\text{(5.3.d)} \qquad \sum_{i=1}^{p} \mathrm{ord}_{x_i}(g_i)\,\mathrm{div}_{\mathcal{L}|_{V_i}}(s_i) - \mathrm{ord}_{x_i}(f_i)\,\mathrm{div}_{\mathcal{M}|_{V_i}}(t_i) = \sum_{i=1}^{p} \mathrm{div} \circ \partial_{A_i}(f_i, g_i).$$

To do so, we compare the coefficients at a point $y \in X$ of codimension two. Let $A = \mathcal{O}_{X,y}$ and $\mathfrak{p}_i \in \mathrm{Spec}\, A$ the primes corresponding to x_i, for $i = 1, \cdots, p$. Let σ, resp. τ, be a generator of the $\mathcal{O}_{X,y}$-module $\mathcal{L}_y$, resp. $\mathcal{M}_y$, and $f = s/\sigma \in k(X)^\times$, resp. $g = t/\tau \in k(X)^\times$. Let $\mathfrak{p}$ be a prime of height one in A corresponding to a point $x \in X$. Then σ, resp. τ, is a generator of the $A_{\mathfrak{p}}$-module $\mathcal{L}_x$, resp. $\mathcal{M}_x$, hence $\mathrm{ord}_{\mathcal{L},x}(s) = \mathrm{ord}_{A_{\mathfrak{p}}}(f)$, resp. $\mathrm{ord}_{\mathcal{M},x}(s) = \mathrm{ord}_{A_{\mathfrak{p}}}(g)$. These integer vanish unless $\mathfrak{p} \in \{\mathfrak{p}_1, \cdots, \mathfrak{p}_p\}$. Then by Theorem 5.2.1, we have

$$0 = \sum_{\text{height }\mathfrak{p}=1} \mathrm{ord}_{A/\mathfrak{p}} \circ \partial_{A_{\mathfrak{p}}}(f, g) = \sum_{i=1}^{p} \mathrm{ord}_{A/\mathfrak{p}_i} \circ \partial_{A_i}(f, g)$$

Let now $a_i, b_i \in (A_i)^\times$ be such that $a_i s_i = \sigma$ and $b_i t_i = \tau$. Then $f = a_i^{-1} f_i \in k(X)^\times$ and $g = b_i^{-1} g_i \in k(X)^\times$. Thus

$$\begin{aligned}
0 &= \sum_{i=1}^{p} \mathrm{ord}_{A/\mathfrak{p}_i} \circ \partial_{A_i}(a_i^{-1} f_i, b_i^{-1} g_i) \\
&= \sum_{i=1}^{p} \mathrm{ord}_{A/\mathfrak{p}_i} \left(\partial_{A_i}(f_i, g_i) \cdot a_i^{-\mathrm{ord}_{x_i}(g_i)} \cdot b_i^{\mathrm{ord}_{x_i}(f_i)} \right) \\
&= \sum_{i=1}^{p} \mathrm{ord}_{A/\mathfrak{p}_i} \circ \partial_{A_i}(f_i, g_i) - \mathrm{ord}_A(a_i)\,\mathrm{ord}_{x_i}(g_i) + \mathrm{ord}_A(b_i)\,\mathrm{ord}_{x_i}(f_i)
\end{aligned}$$

To obtain (5.3.d), observe that the coefficients at y of $\mathrm{div}_{\mathcal{L}|_{V_i}}(s_i)$ and $\mathrm{div}_{\mathcal{M}|_{V_i}}(t_i)$ are respectively $\mathrm{ord}_{A/\mathfrak{p}_i}(a_i)$ and $\mathrm{ord}_{A/\mathfrak{p}_i}(b_i)$. This proves (i).

Let us now prove (ii). One reduces as above to the case when X is normal using additionally Proposition 4.3.2 and Proposition 4.4.4. We then set $\mathcal{L} = \mathcal{O}(D)$ and $s = 1_D$, and proceed as above with the following difference: when $i \in \{1, \cdots, p\}$ is such that $x_i \notin D$, we choose $s_i = 1_{D \cap V_i}$. This ensures that $f_i = 1$ for such i, so that $\operatorname{div} \circ \partial_{A_i}(f_i, g_i) = 0$. Thus the right hand side of (5.3.d) actually lies in $\mathcal{R}(D)$. The class in $\mathrm{CH}(D)$ of the left hand side is

$$D \cdot \operatorname{div}_{\mathcal{M}}(t) - c_1(\mathcal{M})[D],$$

and (ii) follows.

The proof of (iii) is similar. We may as above assume that X is normal. We set $\mathcal{L} = \mathcal{O}(D), s = 1_D$ and $\mathcal{M} = \mathcal{O}(E), t = 1_E$. When $x_i \notin D$, resp. $x_i \notin E$, we choose $s_i = 1_{D \cap V_i}$, resp. $t_i = 1_{E \cap V_i}$. Then when $x_i \notin D \cap E$ we have either $f_i = 1$ or $g_i = 1$, so that $\partial_{A_i}(f_i, g_i) = 1$, and $\operatorname{div} \circ \partial_{A_i}(f_i, g_i) = 0$. Thus the right hand side of (5.3.d) lies in $\mathcal{R}(D \cap E)$, while the class of the left hand side is

$$D \cdot [E] - E \cdot [D],$$

proving (iii). □

COROLLARY 5.3.2. *Let X be a variety and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then we have $c_1(\mathcal{L})\,\mathcal{R}(X) \subset \mathcal{R}(X)$, which gives a morphism*

$$c_1(\mathcal{L})\colon \mathrm{CH}_\bullet(X) \to \mathrm{CH}_{\bullet-1}(X).$$

PROOF. Let V be an integral closed subscheme of X, and $\varphi \in k(V)^\times$. Let s be a regular meromorphic section of $\mathcal{L}|_V$. We view φ as a regular meromorphic section of $\mathcal{O}_V$, and apply Theorem 5.3.1 (i). We obtain, in $\mathrm{CH}(V)$

$$c_1(\mathcal{L}) \circ \operatorname{div} \varphi = c_1(\mathcal{O}_X) \circ \operatorname{div}_{\mathcal{L}}(s)$$

which vanishes by Proposition 4.2.1 (iii). □

COROLLARY 5.3.3. *Let X be a variety and $\mathcal{L}, \mathcal{M}$ an invertible $\mathcal{O}_X$-modules. Then*

$$c_1(\mathcal{L}) \circ c_1(\mathcal{M}) = c_1(\mathcal{M}) \circ c_1(\mathcal{L})\colon \mathrm{CH}_\bullet(X) \to \mathrm{CH}_{\bullet-2}(X)$$

PROOF. We may assume that X is integral and prove that the two morphisms have the same effect on the class $[X]$. Choose a regular meromorphic section s of $\mathcal{L}$, resp. t of $\mathcal{M}$. Then we have in $\mathrm{CH}(X)$ by Theorem 5.3.1 (i):

$$c_1(\mathcal{L}) \circ c_1(\mathcal{M})[X] = c_1(\mathcal{L}) \circ \operatorname{div}_{\mathcal{M}}(t) = c_1(\mathcal{M}) \circ \operatorname{div}_{\mathcal{L}}(s) = c_1(\mathcal{M}) \circ c_1(\mathcal{L})[X]. \quad □$$

4. The Gysin map for divisors

DEFINITION 5.4.1. Let $i\colon D \to X$ be an effective Cartier divisor. We define a group homomorphism

$$\begin{array}{cccc} i^*\colon & \mathcal{Z}_\bullet(X) & \to & \mathrm{CH}_{\bullet-1}(D) \\ & \alpha & \mapsto & D \cdot \alpha. \end{array}$$

COROLLARY 5.4.2 (of Theorem 5.3.1). *We have $i^*\,\mathcal{R}(X) \subset \mathcal{R}(D)$.*

PROOF. Let V be an integral closed subscheme of X, and $\varphi \in k(V)^\times$. If $V \subset D$, then by definition

$$i^* \circ \operatorname{div} \varphi = c_1(\mathcal{O}(D)) \circ \operatorname{div} \varphi \in \mathrm{CH}(D),$$

which vanishes by Corollary 5.3.2. If $V \not\subset D$, then by Theorem 5.3.1 (ii) applied to the variety V

$$D \cdot \operatorname{div} \varphi = c_1(\mathcal{O}_D)[D] \in \mathrm{CH}(D)$$

which vanishes by Proposition 4.2.1 (iii). □

Definition 5.4.3. The induced morphism $i^*\colon \mathrm{CH}_\bullet(X) \to \mathrm{CH}_{\bullet-1}(D)$ is called the *Gysin map.*

Lemma 5.4.4. *Let $i\colon D \to X$ be an effective Cartier divisor. Then*

(i) $i^* \circ i_* = c_1(\mathcal{O}(D)|_D)\colon \mathrm{CH}(D) \to \mathrm{CH}(D)$.
(ii) $i_* \circ i^* = c_1(\mathcal{O}(D))\colon \mathrm{CH}(X) \to \mathrm{CH}(X)$.

Proof. The first statement follows from Definition 4.4.1, and the second from Lemma 4.4.3. □

Lemma 5.4.5. *Let $i\colon D \to X$ be an effective Cartier divisor. If X is equidimensional, then $i^*[X] = [D]$.*

Proof. This is a reformulation of Lemma 4.4.2. □

Proposition 5.4.6. *Consider a cartesian square*

$$\begin{CD} E @>{j}>> Y \\ @V{g}VV @VV{f}V \\ D @>{i}>> X \end{CD}$$

Assume that i and j are both effective Cartier divisors.

(i) If f is proper, then

$$i^* \circ f_* = g_* \circ j^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(D).$$

(ii) If f is flat and has a relative dimension, then

$$f^* \circ i^* = j^* \circ g^*\colon \mathrm{CH}(X) \to \mathrm{CH}(E).$$

Proof. This follows from Proposition 4.4.4 and Proposition 4.4.5. □

CHAPTER 6

Chow groups of bundles

1. Vector bundles, projective bundles

In this section X is a variety.

Vector bundles. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. We consider the graded $\mathcal{O}_X$-algebra

$$\mathcal{S}(\mathcal{E}) = \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}^\vee)$$

whose component of degree n is the n-th symmetric power of the dual $\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$ of $\mathcal{E}$. Then $\mathcal{S}(\mathcal{E})$ is quasi-coherent as an $\mathcal{O}_X$-module, and finitely generated as an $\mathcal{O}_X$-algebra. The *vector bundle associated with* $\mathcal{E}$ is the variety

$$\mathbb{V}(\mathcal{E}) = \mathrm{Spec}_X\, \mathcal{S}(\mathcal{E}).$$

The morphism $\mathbb{V}(\mathcal{E}) \to X$ is affine and flat of relative dimension r. The *rank of* $\mathbb{V}(\mathcal{E})$ is r. The morphism of $\mathcal{O}_X$-algebras $\mathcal{O}_X \to \mathcal{S}(\mathcal{E})$ has a section, which induces a closed immersion $X \to \mathbb{V}(\mathcal{E})$ called the *zero section*. When $\mathcal{E}$ is free, then $\mathbb{V}(\mathcal{E}) \simeq \mathbb{A}^r_X$. A vector bundle of rank one will be called a line bundle. Note that $\mathcal{E}$ can be recovered as the sheaf of sections of the morphism $\mathbb{V}(\mathcal{E}) \to X$. A morphism of locally free $\mathcal{O}_X$-modules $\mathcal{E} \to \mathcal{F}$ induces a morphism $\mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$ of schemes over X, giving an equivalence between the categories of locally free modules and vector bundles. This will allows us to talk about exact sequences of vector bundles for instance. We will write 0 for the vector bundle $\mathbb{V}(0) = X$, and 1 for $\mathbb{V}(\mathcal{O}_X) = X \times \mathbb{A}^1$.

Projective bundles. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r, and $E = \mathbb{V}(\mathcal{E})$. The *projective bundle associated with* $\mathcal{E}$ *(or* E*)* is the variety

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}(E) = \mathrm{Proj}_X\, \mathcal{S}(\mathcal{E}),$$

together with a morphism $p\colon \mathbb{P}(\mathcal{E}) \to X$. The variety $\mathbb{P}(\mathcal{E})$ is equipped with an invertible module $\mathcal{O}(1)$, corresponding to the graded $\mathcal{O}_X$-module $\mathcal{S}(\mathcal{E})(1)$, whose component of degree n is $\mathrm{Sym}^{n+1}_{\mathcal{O}_X}(\mathcal{E}^\vee)$. Observe that the natural morphisms

$$\mathcal{E}^\vee \otimes \mathrm{Sym}^n_{\mathcal{O}_X}(\mathcal{E}^\vee) \to \mathrm{Sym}^{n+1}_{\mathcal{O}_X}(\mathcal{E}^\vee)$$

induce a surjection $p^*\mathcal{E}^\vee \to \mathcal{O}(1)$. In other words, we may view $\mathcal{O}(-1)$ as a sub-bundle of $p^*\mathcal{E}$.

When $\mathcal{E} = 0$, then $\mathbb{P}(\mathcal{E}) = \varnothing$. When $r = 1$, the morphism $p\colon \mathbb{P}(\mathcal{E}) \to X$ is an isomorphism; in addition the surjection $p^*\mathcal{E}^\vee \to \mathcal{O}(1)$ has invertible modules as source and target, hence is an isomorphism. If $r > 0$, the morphism $\mathbb{P}(\mathcal{E}) \to X$ is proper, and flat of relative dimension $r-1$ (but has no canonical section). A injective morphism $\mathcal{E} \to \mathcal{F}$ of locally free $\mathcal{O}_X$-modules induces a surjection $\mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{E})$ of $\mathcal{O}_X$-algebras, and therefore a closed immersion $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{F})$ of schemes over X.

When $E = \mathbb{V}(\mathcal{E})$ is a vector bundle, we denote by $E \oplus 1$ the vector bundle $\mathbb{V}(\mathcal{E} \oplus \mathcal{O}_X)$. The morphism $\mathcal{E} \subset \mathcal{E} \oplus \mathcal{O}_X$ induces a closed immersion $\mathbb{P}(E) \to \mathbb{P}(E \oplus 1)$ (over X). We claim that the complement is the open immersion $E \to \mathbb{P}(E \oplus 1)$ (over X). Indeed $\mathcal{S}(\mathcal{E} \oplus 1) = \mathcal{S}(\mathcal{E})[t]$ for a global section t of degree one, and the closed immersion $\mathbb{P}(E) \to \mathbb{P}(E \oplus 1)$ is the effective Cartier divisor corresponding to the graded ideal generated by t. Its open complement is the relative spectrum over X of the algebra $\mathcal{S}(\mathcal{E})[t]_{(t)}$, consisting of degree one elements in the algebra $\mathcal{S}(\mathcal{E})[t]$ with powers of t inverted. But the $\mathcal{O}_X$-algebra $\mathcal{S}(\mathcal{E})[t]_{(t)}$ is isomorphic to $\mathcal{S}(\mathcal{E})$, as required.

Consider an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ has rank one. Let $p\colon \mathbb{P}(\mathcal{E}) \to X$ be the morphism. Then we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \operatorname{Sym}^{n-1}_{\mathcal{O}_X}(\mathcal{E}^\vee) \to \operatorname{Sym}^{n}_{\mathcal{O}_X}(\mathcal{E}^\vee) \to \operatorname{Sym}^{n}_{\mathcal{O}_X}(\mathcal{F}^\vee) \to 0$$

(the exactness may be checked locally, where $\mathcal{E} = \mathcal{L} \oplus \mathcal{F}$). Thus $\mathcal{L}^\vee(-1) \otimes_{\mathcal{O}_X} \mathcal{S}(\mathcal{E})$ a graded ideal of $\mathcal{S}(\mathcal{E})$, and the corresponding closed subscheme is the effective Cartier divisor $\mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ whose invertible module $\mathcal{O}(\mathbb{P}(\mathcal{F}))$ is isomorphic to $(p^*\mathcal{L})(1)$.

2. Segre classes

DEFINITION 6.2.1. Let E be a vector bundle of rank r on a variety X, and write $p\colon \mathbb{P}(E \oplus 1) \to X$ for the projective bundle. For $i \in \mathbb{Z}$, we define the *i-the Segre class*

$$s_i(E) = p_* \circ c_1(\mathcal{O}(1))^{r+i} \circ p^*\colon \operatorname{CH}_\bullet(X) \to \operatorname{CH}_{\bullet - i}(X).$$

Here we have used the convention that $c_1(\mathcal{O}(1))^n = 0$ for $n < 0$. Observe that $s_i(E) = 0$ when $i \notin \{-r, \cdots, \dim X\}$. We will write

$$s(E) = \sum_{i \in \mathbb{Z}} s_i(E).$$

LEMMA 6.2.2. *We have* $s(0) = \mathrm{id}$.

PROOF. Indeed when $E = 0$, then the projection $\mathbb{P}(E \oplus 1) \to X$ is an isomorphism and the line bundle $\mathcal{O}(1)$ is trivial. □

PROPOSITION 6.2.3. *Let $f\colon Y \to X$ be a morphism of varieties, and E a vector bundle on X.*

(i) If f is proper, then

$$s(E) \circ f_* = f_* \circ s(f^*E)\colon \operatorname{CH}(Y) \to \operatorname{CH}(X).$$

(ii) If f is flat and has a relative dimension, then

$$s(f^*E) \circ f^* = f^* \circ s(E)\colon \operatorname{CH}(X) \to \operatorname{CH}(Y).$$

PROOF. Consider the cartesian square

$$\begin{array}{ccc} \mathbb{P}(f^*E \oplus 1) & \xrightarrow{g} & \mathbb{P}(E \oplus 1) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

We have $g^*\mathcal{O}(1) = \mathcal{O}(1)$. Let r be the rank of E.

(i): If f is proper then so is g, and we have, for any i

$$\begin{aligned}
s_i(E) \circ f_* &= p_* \circ c_1(\mathcal{O}(1))^{r+i} \circ p^* \circ f_* \\
&= p_* \circ c_1(\mathcal{O}(1))^{r+i} \circ g_* \circ q^* && \text{by Proposition 1.5.9} \\
&= p_* \circ g_* \circ c_1(\mathcal{O}(1))^{r+i} \circ q^* && \text{by Proposition 4.2.2} \\
&= f_* \circ q_* \circ c_1(\mathcal{O}(1))^{r+i} \circ q^* && \text{by Lemma 1.2.6} \\
&= f_* \circ s_i(f^*E).
\end{aligned}$$

(ii): If f is flat and has a relative dimension, then the same is true for g, and we have, for any i

$$\begin{aligned}
s_i(f^*E) \circ f^* &= p_* \circ c_1(\mathcal{O}(1))^{r+i} \circ p^* \circ f^* \\
&= p_* \circ c_1(\mathcal{O}(1))^{r+i} \circ g^* \circ q^* && \text{by Proposition 1.5.8} \\
&= p_* \circ g^* \circ c_1(\mathcal{O}(1))^{r+i} \circ q^* && \text{by Proposition 4.2.3} \\
&= f^* \circ q_* \circ c_1(\mathcal{O}(1))^{r+i} \circ q^* && \text{by Proposition 1.5.9} \\
&= f^* \circ s_i(E). && \square
\end{aligned}$$

LEMMA 6.2.4. *Let $E \to X$ be a vector bundle. Then $s_i(E) = 0$ for $i < 0$.*

PROOF. Let $v \colon V \to X$ be the closed immersion of an integral closed subscheme. By Proposition 6.2.3 (i), we have $s(E)[V] = v_* \circ s(E|_V)[V]$. But $s(E|_V)[V]$ belongs to $\mathrm{CH}_{\dim V - i}(V)$, a group which vanishes when $i < 0$. $\square$

LEMMA 6.2.5. *Let E and F be two isomorphic vector bundles over X. Then*

$$s(E) = s(F).$$

PROOF. Let r be the rank of E and F, and $p \colon \mathbb{P}(E \oplus 1) \to X$ and $q \colon \mathbb{P}(F \oplus 1) \to X$ the projective bundles. We have an isomorphism $\varphi \colon \mathbb{P}(E \oplus 1) \to \mathbb{P}(F \oplus 1)$ such that $\varphi^*\mathcal{O}(1) = \mathcal{O}(1)$ and $q \circ \varphi = p$. In particular $\varphi_* \circ \varphi^* = \mathrm{id}$, and we have for any i

$$\begin{aligned}
s_i(E) &= p_* \circ c_1(\mathcal{O}(1))^{r+i} \circ p^* \\
&= q_* \circ \varphi_* \circ c_1(\mathcal{O}(1))^{r+i} \circ \varphi^* \circ q^* && \text{by (1.2.6), (1.5.8)} \\
&= q_* \circ \varphi_* \circ \varphi^* \circ c_1(\mathcal{O}(1))^{r+i} \circ q^* && \text{by (4.2.3)} \\
&= q_* \circ c_1(\mathcal{O}(1))^{r+i} \circ q^* \\
&= s_i(F).
\end{aligned}$$

$\square$

PROPOSITION 6.2.6. *Let E and F be two vector bundles on X. Then for any i, j*

$$s_i(E) \circ s_j(F) = s_j(F) \circ s_i(E).$$

PROOF. Consider the cartesian square

$$\begin{CD}
Q @>{q'}>> \mathbb{P}(E \oplus 1) \\
@V{p'}VV @VV{p}V \\
\mathbb{P}(F \oplus 1) @>{q}>> X
\end{CD}$$

Let r, resp. s, be the rank of E, resp. F. Then

$$\begin{aligned}
s_i(E)\circ s_j(F) &= p_*\circ c_1(\mathcal{O}(1))^{r+i}\circ p^*\circ q_*\circ c_1(\mathcal{O}(1))^{s+j}\circ q^* \\
&= p_*\circ c_1(\mathcal{O}(1))^{r+i}\circ q'_*\circ p'^*\circ c_1(\mathcal{O}(1))^{s+j}\circ q^* && \text{by (1.5.9)}\\
&= p_*\circ q'_*\circ c_1(q'^*\mathcal{O}(1))^{r+i}\circ c_1(p'^*\mathcal{O}(1))^{s+j}p'^*\circ q^* && \text{by (4.2.2), (4.2.3)}\\
&= p_*\circ q'_*\circ c_1(p'^*\mathcal{O}(1))^{s+j}\circ c_1(q'^*\mathcal{O}(1))^{r+i}p'^*\circ q^* && \text{by (5.3.3)}\\
&= q_*\circ p'_*\circ c_1(p'^*\mathcal{O}(1))^{s+j}\circ c_1(q'^*\mathcal{O}(1))^{r+i}\circ q'^*\circ p^* && \text{by (1.2.6), (1.5.8)}\\
&= q_*\circ c_1(p'^*\mathcal{O}(1))^{s+j}\circ p'_*\circ q'^*\circ c_1(q'^*\mathcal{O}(1))^{r+i}\circ p^* && \text{by (4.2.2), (4.2.3)}\\
&= q_*\circ c_1(p'^*\mathcal{O}(1))^{s+j}\circ p^*\circ q_*\circ c_1(q'^*\mathcal{O}(1))^{r+i}\circ p^* && \text{by (1.5.9)}\\
&= s_j(F)\circ s_i(E).
\end{aligned}$$

□

LEMMA 6.2.7. *Let E be a vector bundle over X. Denote by $j\colon \mathbb{P}(E)\to\mathbb{P}(E\oplus 1)$ be the induced closed immersion, and consider the projective bundles $p\colon \mathbb{P}(E\oplus 1)\to X$ and $q=p\circ j\colon \mathbb{P}(E)\to X$. Then, for any $n\geq 0$, we have*

$$j_*\circ c_1(\mathcal{O}(1))^n\circ q^* = c_1(\mathcal{O}(1))^{n+1}\circ p^*.$$

PROOF. Let V be an integral closed subscheme of X. Then the closed immersions $\mathbb{P}(E|_V)\to\mathbb{P}(E)$ and $\mathbb{P}(E\oplus 1|_V)\to\mathbb{P}(E\oplus 1)$ are compatible with the line bundles $\mathcal{O}(1)$. Replacing X by V, it will suffice to prove that

$$j_*\circ c_1(\mathcal{O}(1))^n[\mathbb{P}(E)] = c_1(\mathcal{O}(1))^{n+1}[\mathbb{P}(E\oplus 1)].$$

The closed immersion $\mathbb{P}(E)\to\mathbb{P}(E\oplus 1)$ is an effective Cartier divisor whose line bundle $\mathcal{O}(\mathbb{P}(E))$ is isomorphic to $\mathcal{O}(1)$. Since $j^*\mathcal{O}(1)=\mathcal{O}(1)$, it follows from Proposition 4.2.2 that $j_*\circ c_1(\mathcal{O}(1))^n[\mathbb{P}(E)] = c_1(\mathcal{O}(1))^n\circ j_*[\mathbb{P}(E)]$. But $j_*[\mathbb{P}(E)] = c_1(\mathcal{O}(1))[\mathbb{P}(E\oplus 1)]$ by Lemma 4.3.1. □

LEMMA 6.2.8. *Let $E\to X$ be a vector bundle of rank r.*

(i) Let $q\colon \mathbb{P}(E)\to X$ be the projective bundle. If $r>0$, then

$$s_i(E) = q_*\circ c_1(\mathcal{O}(1))^{r-1+i}\circ q^*.$$

(ii) We have $s(E\oplus 1) = s(E)$.

PROOF. Let $p\colon \mathbb{P}(E\oplus 1)\to X$ be the projective bundle.

(i): We apply Lemma 6.2.7. Then we have, for any $i\geq 1-r$

$$\begin{aligned}
s_i(E) &= p_*\circ c_1(\mathcal{O}(1))^{r+i}\circ p^*\\
&= p_*\circ j_*\circ c_1(\mathcal{O}(1))^{r-1+i}\circ q^*\\
&= q_*\circ c_1(\mathcal{O}(1))^{r-1+i}\circ q^*.
\end{aligned}$$

This formula also holds in case $i<1-r\leq 0$, by Lemma 6.2.4.

(ii): Applying (i) to the bundle $E\oplus 1$, we have, for any i

$$s_i(E\oplus 1) = p_*\circ c_1(\mathcal{O}(1))^{(r+1)-1-i}\circ p^* = s_i(E).$$

□

When $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module and $L\to X$ the corresponding line bundle, we will write $c_1(L)$ for $c_1(\mathcal{L})$.

LEMMA 6.2.9. *Let $L\to X$ be a line bundle. Then, for every i*

$$s_i(L) = (-c_1(L))^i.$$

Proof. The morphism $q\colon \mathbb{P}(L) \to X$ is an isomorphism, and $\mathcal{O}(1) = q^*\mathcal{L}^\vee$, where $\mathcal{L}$ is the $\mathcal{O}_X$-module of sections of L. In particular $q_* \circ q^* = \mathrm{id}$. We have, for any i,

$$\begin{aligned} s_i(L) &= q_* \circ c_1(\mathcal{O}(1))^i \circ q^* && \text{by Lemma 6.2.8 (i)} \\ &= q_* \circ c_1(q^*\mathcal{L}^\vee)^i \circ q^* \\ &= q_* \circ q^* \circ c_1(\mathcal{L}^\vee)^i && \text{by Proposition 4.2.3} \\ &= q_* \circ q^* \circ (-c_1(\mathcal{L}))^i && \text{by Proposition 4.2.1 (ii), (iii)} \\ &= (-c_1(\mathcal{L}))^i. && \square \end{aligned}$$

Lemma 6.2.10. *Let $E \to X$ be a vector bundle. Then $s_0(E) = \mathrm{id}$.*

Proof. When $r = 0$, then $\mathbb{P}(E \oplus 1) \to X$ is an isomorphism, and $s_0(E) = \mathrm{id}$. Assume that $r > 0$. By Proposition 6.2.3 (i), it will suffice to assume that X is integral, and prove that $s_0(E)[X] = [X]$. As $s_0(E)[X]$ belongs to $\mathrm{CH}_{\dim X}(X)$, the free abelian group generated by $[X]$, we may write $s_0(E)[X] = m[X]$ for some integer m. To prove that $m = 1$, we may restrict to an open non-empty subscheme of X, and assume that $E = E' \oplus 1$ for some vector bundle E' on X. Then the statement follows from Lemma 6.2.8 (ii) and induction on r. ☐

Proposition 6.2.11. *Let $E \to X$ be a vector bundle of rank $r > 0$, and consider the projective bundle $q\colon \mathbb{P}(E) \to X$. Then the pull-back*

$$q^*\colon\ \mathrm{CH}(X) \to \mathrm{CH}(\mathbb{P}(E))$$

is a split monomorphism.

Proof. In view of Lemma 6.2.8 (i) and Lemma 6.2.10, the splitting is given by $q_* \circ c_1(\mathcal{O}(1))^{r-1}$. ☐

Proposition 6.2.12. *Consider an exact sequence of vector bundles on X*

$$0 \to E \to F \to G \to 0.$$

Then we have

$$s(E) \circ s(G) = s(F).$$

Proof. Let r be the rank of F. First assume that G is a line bundle (so that in particular $r \geq 1$). Let $q\colon \mathbb{P}(E \oplus 1) \to X$ and $p\colon \mathbb{P}(F \oplus 1) \to X$ be the projective bundles, and $j\colon \mathbb{P}(E \oplus 1) \to \mathbb{P}(F \oplus 1)$ the closed immersion. We claim that

$$j_* \circ q^* = \big(c_1(\mathcal{O}(1)) + c_1(p^*G)\big) \circ p^*. \tag{6.2.e}$$

To see this, it suffices to prove that the two morphisms have the same effect on the class $[V]$ of an integral closed subscheme V of X. To do so, we may assume that $V = X$. Since j is an effective Cartier divisor whose invertible module $\mathcal{O}(\mathbb{P}(E \oplus 1))$ is isomorphic to $p^*\mathcal{G}(1)$ (where $\mathcal{G}$ is the $\mathcal{O}_X$-module of sections of G), we have

$$\begin{aligned} j_* \circ q^*[X] &= j_*[\mathbb{P}(E)] \\ &= c_1(p^*\mathcal{G}(1))[\mathbb{P}(F)] && \text{by (4.3.1)} \\ &= (c_1(\mathcal{O}(1)) + c_1(p^*\mathcal{G}))[\mathbb{P}(F)] && \text{by (4.2.1) (i) (ii)} \\ &= (c_1(\mathcal{O}(1)) + c_1(p^*\mathcal{G})) \circ p^*[X], \end{aligned}$$

which proves the claim. Since $j^*\mathcal{O}(1) = \mathcal{O}(1)$, we have for any $i \geq 0$

$$\begin{aligned}
s_i(E) &= q_* \circ c_1(\mathcal{O}(1))^{r-1+i} \circ q^* \\
&= p_* \circ j_* \circ c_1(\mathcal{O}(1))^{r-1+i} \circ q^* && \text{by (1.2.6)} \\
&= p_* \circ c_1(\mathcal{O}(1))^{r-1+i} \circ j_* \circ q^* && \text{by (4.2.2)} \\
&= p_* \circ c_1(\mathcal{O}(1))^{r-1+i} \circ \big(c_1(\mathcal{O}(1)) + c_1(p^*G)\big) \circ p^* && \text{by (6.2.e)} \\
&= s_i(F) + s_{i-1}(F) \circ c_1(G) && \text{by (4.2.3).}
\end{aligned}$$

This formula also holds for $i < 0$ by Lemma 6.2.4. It follows that

$$s(E) = s(F) \circ (\mathrm{id} + c_1(G)).$$

By Lemma 6.2.9, and since $c_1(G)^n = 0$ for $n > \dim X$, we have

$$(\mathrm{id} + c_1(G)) \circ s(G) = (\mathrm{id} + c_1(G) \circ \sum_{i \in \mathbb{Z}} (-c_1(G))^i = \mathrm{id},$$

and the statement follows in the case when G is a line bundle.

We prove the statement when G is arbitrary for all varieties X simultaneously, by induction on r. If $r = 0$ the statement is true by Lemma 6.2.2, since $E = F = G = 0$. Assume that $r > 0$. If $G = 0$, then E and F are isomorphic, and the statement follows from Lemma 6.2.2 and Lemma 6.2.5. Thus we may assume that G has rank > 0, and let $f \colon \mathbb{P}(G^\vee) \to X$ be the projective bundle. Let $L \to \mathbb{P}(G^\vee)$ be the line bundle whose module of sections is $\mathcal{O}(1)$. Recalling that $\mathcal{O}(1)$ is canonically a quotient of $p^*\mathcal{G}^\vee$, we obtain exact sequences of vector bundles over $\mathbb{P}(G^\vee)$

$$0 \to H \to f^*G \to L \to 0$$

$$0 \to M \to f^*F \to L \to 0$$

$$0 \to f^*E \to M \to H \to 0$$

By the induction hypothesis, since the rank of M is $r - 1$, we have

$$s(M) = s(f^*E) \circ s(H)$$

and by the case of a line bundle treated above

$$s(f^*G) = s(H) \circ s(L) \quad \text{and} \quad s(f^*F) = s(M) \circ s(L).$$

It follows that

$$s(f^*F) = s(f^*E) \circ s(f^*G).$$

Therefore, by Proposition 6.2.3 (ii)

$$f^* \circ s(F) = s(f^*F) \circ f^* = s(f^*E) \circ s(f^*G) \circ f^* = f^* \circ s(E) \circ s(G).$$

We conclude using the injectivity of $f^* \colon \mathrm{CH}(X) \to \mathrm{CH}(\mathbb{P}(G^\vee))$ (Proposition 6.2.11), since $G^\vee$ has the same rank as G, which is > 0. □

3. Homotopy invariance and projective bundle theorem

Proposition 6.3.1. *Let $v\colon E \to X$ be a vector bundle. Then the pull-back*

$$v^*\colon \mathrm{CH}(X) \to \mathrm{CH}(E)$$

is surjective.

Proof. — *Case $E = \mathbb{A}^1 \times X$ and v is the second projection* : Let W be an integral closed subscheme of $\mathbb{A}^1 \times X$, and V the closure of its image in X. To prove that $[W]$ is in the image of $\mathrm{CH}(X) \to \mathrm{CH}(\mathbb{A}^1 \times X)$, it will suffice to prove that $[W]$ is in the image of $\mathrm{CH}(V) \to \mathrm{CH}(\mathbb{A}^1 \times V)$, and we may therefore assume that $W \to X$ is dominant, and that X is integral. Then $\dim W \geq \dim X$; if $\dim W = \dim X + 1$, then $W = \mathbb{A}^1 \times X$, and $[W] = v^*[X]$. Thus we may assume that $\dim W = \dim X$. We write $K = k(X)$ for the function field of X. The generic fiber $W_K = W \times_X \operatorname{Spec} K$ is a closed subscheme of $\mathbb{A}^1_K$, hence is defined by a single polynomial $p \in K[t]$. Since $W_K \neq \mathbb{A}^1_K$ we have $p \neq 0$, and thus $W_K \to \mathbb{A}^1_K$ is an effective Cartier divisor. Then by Lemma 2.1.5

$$[W_K] = \operatorname{div} p \in \mathcal{Z}(\mathbb{A}^1_K).$$

We may view p as a nonzero element $\varphi \in k(\mathbb{A}^1 \times X) = K(t)$. For any integral closed subscheme Z of $\mathbb{A}^1 \times X$ dominating X, the coefficient at $[Z]$ of $[W] - \operatorname{div} \varphi \in \mathcal{Z}(\mathbb{A}^1 \times X)$ coincides with the coefficient at $[Z \times_X \operatorname{Spec} K]$ of $[W_K] - \operatorname{div} p \in \mathcal{Z}(\mathbb{A}^1_K)$, which vanishes by construction. Thus the cycle

$$[W] - \operatorname{div} \varphi \in \mathcal{Z}(\mathbb{A}^1 \times X)$$

lies in the subgroup $\mathcal{Z}(\mathbb{A}^1 \times Y)$, for some closed subscheme $Y \neq X$ of X. Thus

$$[W] \in \operatorname{im}(\mathrm{CH}(\mathbb{A}^1 \times Y) \to \mathrm{CH}(\mathbb{A}^1 \times X)).$$

In view of Proposition 1.5.9, we may conclude by noetherian induction, the statement being clear when $X = \varnothing$.

— *Case $E = \mathbb{A}^n \times X$ and v is the second projection* : Then v may be decomposed as a sequence of trivial line bundles, and the statement follows from the case considered above.

— *General case* : We can find a non-empty open subscheme U of X such that the vector bundle $E|_U \to U$ is trivial. Let Y be the closed complement of U, endowed with the reduced scheme structure. Then by Proposition 2.3.1, Proposition 1.5.9 and Proposition 1.5.8, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{CH}(E|_Y) & \longrightarrow & \mathrm{CH}(E) & \longrightarrow & \mathrm{CH}(E|_U) & \longrightarrow & 0 \\
\uparrow {\scriptstyle (v|_Y)^*} & & \uparrow {\scriptstyle v^*} & & \uparrow {\scriptstyle (v|_U)^*} & & \\
\mathrm{CH}(Y) & \longrightarrow & \mathrm{CH}(X) & \longrightarrow & \mathrm{CH}(U) & \longrightarrow & 0
\end{array}$$

Using noetherian induction, we may assume that $(v|_Y)^*$ is surjective. Since $(v|_U)^*$ is surjective by the case treated above, it follows from a diagram chase that v^* is surjective. $\square$

Theorem 6.3.2 (Projective bundle Theorem). *Let $v\colon E \to X$ be a vector bundle of rank r, and $q\colon \mathbb{P}(E) \to X$ the associated projective bundle. Then the morphism*

$$\theta_E\colon \bigoplus_{i=0}^{r-1} \mathrm{CH}(X) \to \mathrm{CH}(\mathbb{P}(E))$$

given by

$$(\alpha_0, \cdots, \alpha_{r-1}) \mapsto \sum_{i=0}^{r-1} c_1(\mathcal{O}(1))^i \circ q^*(\alpha_i)$$

is bijective.

Theorem 6.3.3 (Homotopy invariance). *Let $v\colon E \to X$ be a vector bundle of rank r. Then the pull-back*

$$v^*\colon \mathrm{CH}(X) \to \mathrm{CH}(E)$$

is bijective.

Proof of Theorem 6.3.3 and Theorem 6.3.2. The case $r = 0$ being clear, we assume that $r > 0$. Assume that $\theta_E(\alpha_0, \cdots, \alpha_{r-1}) = 0$, and let l be the largest integer such that $\alpha_l \neq 0$, if it exists. Then we have in $\mathrm{CH}(X)$

$$\begin{aligned}
0 &= q_* \circ c_1(\mathcal{O}(1))^{r-1-l} \circ \theta_E(\alpha_0, \cdots, \alpha_{r-1}) \\
&= \sum_{i=0}^{l} q_* \circ c_1(\mathcal{O}(1))^{r-1-l+i} \circ q^*(\alpha_i) \\
&= \sum_{i=0}^{l} s_{i-l}(\alpha_i) && \text{by (6.2.8) (i)} \\
&= \alpha_l && \text{by (6.2.4) and (6.2.10).}
\end{aligned}$$

Thus an integer l as above does not exist, proving that θ_E is injective.

Let $j\colon \mathbb{P}(E) \to \mathbb{P}(E \oplus 1)$ be the closed embedding, and $u\colon E \to \mathbb{P}(E \oplus 1)$ its open complement. By Lemma 6.2.7, we have

$$j_* \circ \theta_E(\alpha_0, \cdots, \alpha_{r-1}) = \theta_{E\oplus 1}(0, \alpha_0, \cdots, \alpha_{r-1}).$$

In addition, since $\mathcal{O}(1)|_E = u^*\mathcal{O}(1)$ is the trivial line bundle, we have, by Proposition 4.2.3 and Proposition 4.2.1 (iii)

$$u^* \circ c_1(\mathcal{O}(1)) = c_1(\mathcal{O}(1)|_E) \circ u^* = 0.$$

Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{CH}(\mathbb{P}(E)) & \xrightarrow{j_*} & \mathrm{CH}(\mathbb{P}(E \oplus 1)) & \xrightarrow{u^*} & \mathrm{CH}(E) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \theta_E} & & \uparrow{\scriptstyle \theta_{E\oplus 1}} & & \uparrow{\scriptstyle v^*} & & \\
\bigoplus_{i=0}^{r-1} \mathrm{CH}(X) & \xrightarrow[a]{} & \bigoplus_{i=0}^{r} \mathrm{CH}(X) & \xrightarrow[b]{} & \mathrm{CH}(X) & \longrightarrow & 0
\end{array}$$

where

$$a(\alpha_0, \cdots, \alpha_{r-1}) = (0, \alpha_0, \cdots, \alpha_{r-1}) \;\text{ and }\; b(\alpha_0, \cdots, \alpha_r) = \alpha_0.$$

Therefore Theorem 6.3.3 follows from Theorem 6.3.2. Moreover, using the surjectivity of v^* obtained in Proposition 6.3.1, we deduce from a diagram chase that

$$\theta_E \text{ surjective } \Rightarrow \theta_{E\oplus 1} \text{ surjective .} \tag{6.3.f}$$

We now prove the surjectivity of θ_E for all varieties X simultaneously, by induction on the rank r. For a given $r > 0$ and X, we use noetherian induction. We can find a non-empty open subscheme U of X such that the vector bundle $E|_U$ splits as $E' \oplus 1$, for a vector bundle E' on U. Let Y be the closed complement of U, endowed with the reduced scheme structure. Then by Proposition 2.3.1, Proposition 1.5.9 and Proposition 1.5.8, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{CH}(\mathbb{P}(E|_Y)) & \longrightarrow & \mathrm{CH}(\mathbb{P}(E)) & \longrightarrow & \mathrm{CH}(\mathbb{P}(E|_U)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \theta_{E|_Y}} & & \uparrow{\scriptstyle \theta_E} & & \uparrow{\scriptstyle \theta_{E'\oplus 1}} & & \\
\bigoplus_{i=0}^{r-1} \mathrm{CH}(Y) & \longrightarrow & \bigoplus_{i=0}^{r-1} \mathrm{CH}(X) & \longrightarrow & \bigoplus_{i=0}^{r-1} \mathrm{CH}(U) & \longrightarrow & 0
\end{array}$$

Since the rank of E' is $< r$, the morphism $\theta_{E'}$ is surjective by induction on r, and so is $\theta_{E'\oplus 1}$ by (6.3.f). The morphism $\theta_{E|_Y}$ is surjective by noetherian induction, and it follows from a diagram chase that θ_E is surjective. □

Example 6.3.4.

$$\mathrm{CH}_i(\mathbb{A}^n) = \begin{cases} \mathbb{Z}\cdot[\mathbb{A}^n] & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{CH}_i(\mathbb{P}^n) = \begin{cases} \mathbb{Z}\cdot[\mathbb{P}^i] & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

(Here we $\mathbb{P}^i$ denotes the linear subspace of $\mathbb{P}^n$ of dimension i).

4. Chern classes

Let $E \to X$ be a vector bundle of rank r, and $q\colon \mathbb{P}(E) \to X$ the associated projective bundle. For any $\alpha \in \mathrm{CH}(X)$, by the projective bundle Theorem 6.3.2, there are unique elements

$$c_i(E)(\alpha) \in \mathrm{CH}(X)$$

such that

$$c_0(E)(\alpha) = \alpha \text{ and } c_i(E)(\alpha) = 0 \text{ for } i \notin \{0, \cdots, r\},$$

and

$$0 = \sum_i c_1(\mathcal{O}(1))^{r-i} \circ q^* \circ c_i(E)(\alpha) \in \mathrm{CH}(\mathbb{P}(E)). \tag{6.4.g}$$

This defines group homomorphisms

$$c_i(E)\colon \mathrm{CH}_n(X) \to \mathrm{CH}_{n-i}(X),$$

and we write

$$c(E) = \sum_i c_i(E).$$

Proposition 6.4.1. *Let $E \to X$ be a line bundle with sheaf of sections $\mathcal{E}$. Then the endomorphism $c_1(E)$ defined above coincides with the endomorphism $c_1(\mathcal{E})$ defined in §4.2.*

Proof. Indeed $q\colon \mathbb{P}(E)\to X$ is an isomorphism such that $\mathcal{O}(1)=q^*\mathcal{E}^\vee$. In view of Proposition 4.2.1(ii) and Proposition 4.2.3, we have

$$c_1(\mathcal{O}(1))\circ q^* + q^*\circ c_1(\mathcal{E}) = 0,$$

proving that $c_1(E)=c_1(\mathcal{E})$. □

Proposition 6.4.2. *Let $E\to X$ be a vector bundle of rank r. Then*

$$s(E)\circ c(E) = c(E)\circ s(E) = \mathrm{id}_{\mathrm{CH}(X)}\,.$$

Proof. The case $r=0$ being clear, let us assume that $r>0$ and consider the morphism $q\colon \mathbb{P}(E)\to X$. For any $k\geq 1$, we apply $q_*\circ c_1(\mathcal{O}(1))^{k-1}$ to the relation (6.4.g). Using Lemma 6.2.8 (i), we obtain (since $c_i(E)=0$ for $i<0$)

$$0=\sum_{i\geq 0} q_*\circ c_1(\mathcal{O}(1))^{r+k-1-i}\circ q^*\circ c_i(E) = \sum_{i\geq 0} s_{k-i}(E)\circ c_i(E).$$

On the other hand, in view of Lemma 6.2.4 and Lemma 6.2.10,

$$\sum_{i\geq 0} s_{-i}(E)\circ c_i(E) = s_0(E)\circ c_0(E) = \mathrm{id}\,.$$

Therefore

$$s(E)\circ c(E) = \sum_{k\geq 0}\sum_{i\geq 0} s_{k-i}(E)\circ c_i(E) = \mathrm{id}\,.$$

Since $s_0(E)=\mathrm{id}$, we see that the morphism $s(E)$ is injective. It follows that $c(E)\circ s(E) = \mathrm{id}$. □

Thus the individual Chern classes can be expressed recursively from the Segre classes using the formula

$$c_n(E) = -\sum_{i=0}^{n-1} c_i(E)\circ s_{n-i}(E). \tag{6.4.h}$$

Corollary 6.4.3. *Let E and F be two vector bundles on X. Then for any i,j*

$$c_i(E)\circ c_j(F) = c_j(F)\circ c_i(E).$$

Proof. This follows recursively from (6.4.h) and Proposition 6.2.6. □

Corollary 6.4.4. *Consider an exact sequence of vector bundles on X*

$$0\to E\to F\to G\to 0.$$

Then we have

$$c(E)\circ c(G) = c(F).$$

Proof. This follows from Proposition 6.2.12. □

Corollary 6.4.5. *Let $f\colon Y\to X$ be a morphism, and E be a vector bundle on X.*

(i) If f is proper, then

$$c(E)\circ f_* = f_*\circ c(f^*E)\colon\ \mathrm{CH}(Y)\to\mathrm{CH}(X).$$

(ii) If f is flat and has a relative dimension, then

$$c(f^*E)\circ f^* = f^*\circ c(E)\colon\ \mathrm{CH}(X)\to\mathrm{CH}(Y).$$

Proof. This follows from Corollary 6.4.4, Corollary 6.4.5 and Proposition 6.4.2. □

PROPOSITION 6.4.6. *If the vector bundle $E \to X$ is trivial (i.e. isomorphic to $\mathbb{A}^r_X \to X$) then $c_i(E) = 0$ when $i > 0$.*

PROOF. We prove that $c_i(E)[V] = 0$ when V is an integral closed subscheme of X. In view of Corollary 6.4.5 we may assume that $V = X$. Let $\pi\colon \mathbb{P}(E) = \mathbb{P}^{r-1}_X \to \mathbb{P}^{r-1}$ be the projection. Then in $\mathrm{CH}(\mathbb{P}(E))$

$$c_1(\mathcal{O}(1))^r[\mathbb{P}^{r-1}_X] = c_1(\pi^*\mathcal{O}(1))^r \circ \pi^*[\mathbb{P}^{r-1}] = \pi^* \circ c_1(\mathcal{O}(1))^r[\mathbb{P}^{r-1}]$$

which vanishes, since $c_1(\mathcal{O}(1))^r[\mathbb{P}^{r-1}] \in \mathrm{CH}_{-1}(\mathbb{P}^{r-1}) = 0$. The result follows from the definition of the Chern classes (6.4.g). □

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